Algorithmic Desingularization

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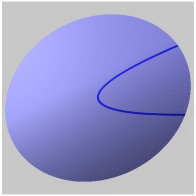
Osaka, June 29th 2010

Introduction by Pictures: Singularities



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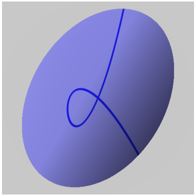
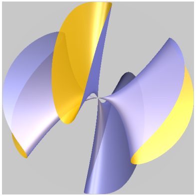
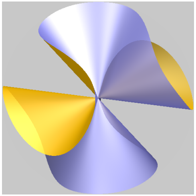


$y^2 - x^2 - x^3 = 0$



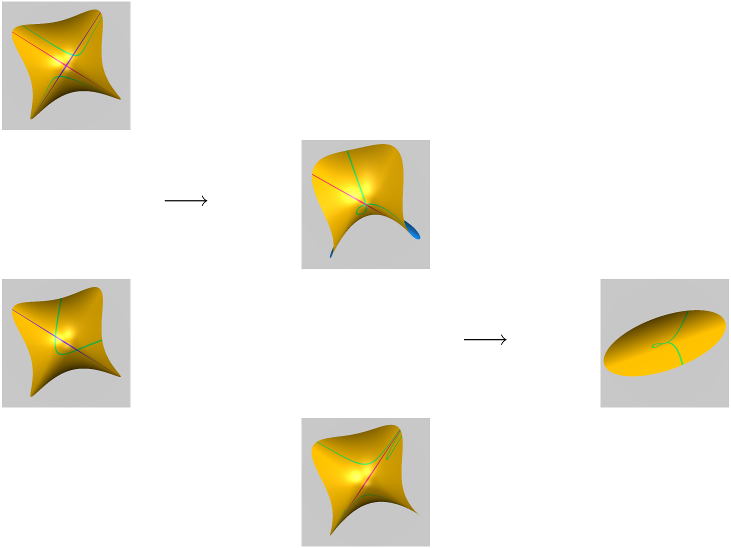
$x^2 + y^4 - z^4 - 3x^2y^2 = 0$



$x^3 + y^3 - z^3 - 11xyz = 0$

Introduction by Pictures: Desingularization

Resolution of $V(y^2 + x^4 - x^5) \subset \mathbb{A}^2$



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such that $\mathrm{Reg}(X) \cong \pi^{-1}(\mathrm{Reg}(X))$.

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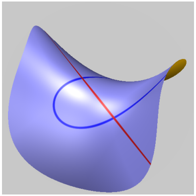
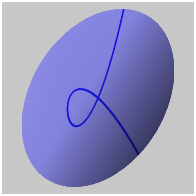
in pictures (just 1 chart):



$\longrightarrow$



Embedded Desingularization

Technical Formulation:

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5. $(W_r, X_r) \longrightarrow (W, X)$ is equivariant under group action

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recent developments:

- algorithmic proofs: Bierstone+Milman; Villamayor; Encinas+Hauser (since 1990s)
- implementations: Bodnar+Schicho; FK+Pfister

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B Finding suitable centers C_i

key difficulty: not all permissible centers improve the situation

Computing a blowing up

X affine variety, its ideal $I_X \subset K[x_1, \ldots, x_n]$
C smooth subvariety of X
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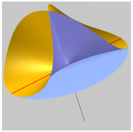
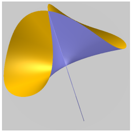
can be computed as preimage of I_X under

$$\begin{array}{rcl} \Phi : K[x_1, \ldots, x_n, y_1, \ldots, y_k] & \longrightarrow & K[x_1, \ldots, x_n, t] \\ x_i & \longmapsto & x_i \\ y_j & \longmapsto & t \cdot f_j \end{array}$$

Some Good and Bad Choices of Centers I



$\xrightarrow{C=\mathbf{0}}$



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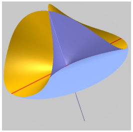

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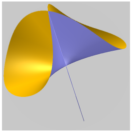

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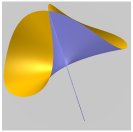

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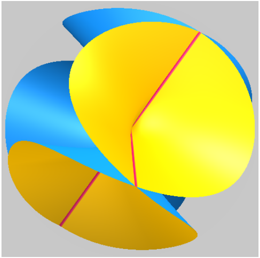

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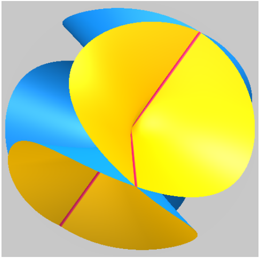

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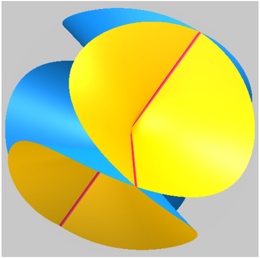

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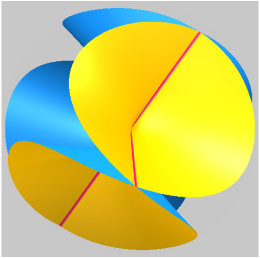

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- decrease of maximal value measures progress of desingularization

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- depending on algorithmic approach inv_n can also have the Hilbert-Samuel function at w as first entry

Variants in SINGULAR

Variants of embedded desingularization:

- variant of Villamayor's Algorithm (available):
  + all dimensions
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- Blanco's variant for binomial ideals (implemented, not yet in distribution):
  + computation of center by combinatorial process
  + faster, total amount of data smaller
- Jung's algorithm for surfaces (implementation FK-Renner):
  - only for surfaces
  + faster, fewer charts

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- Separation of $\mathbb{C}$-irreducible components of exceptional divisors

Applications of desingularization

Given $\pi : W \longrightarrow \mathbb{C}^n$ embedded resolution of $V = f^{-1}(0)$,
E_i irreducible components of $\pi^{-1}(f^{-1}(0))$
N_i multiplicity of E_i in divisor of $f \circ \pi$
$\nu_i - 1$ multiplicity of E_i in divisor of $\pi^*(dx_1 \wedge \cdots \wedge dx_n)$

Currently available:

- intersection form of exceptional curves on desingularized surface
- dual graph of resolution (surface case)
- discrepancies $a_i = \nu_i - N_i$
- topological zeta function (global and local)

$$Z_{top,f}(s) = \sum_I \chi(E_I^\circ) \prod_{i \in I} \frac{1}{N_i s + \nu_i}$$