The moduli space of points on the projective line and quadratic Gröbner bases.

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Let

$$M_n = (\mathbb{P}^1)^n /\!/ \operatorname{Aut}(\mathbb{P}^1).$$

There is a natural embedding

$$M_n \hookrightarrow \mathbb{P}^N,$$

with homogeneous coordinate ring

$$A \cong \mathbb{C}[x_0, \dots, x_N]/I.$$

A is the ring of invariants.

Example

- $M_4 \cong \mathbb{P}^1$
- M_5 del Pezzo surface.
- M_6 is the Segre cubic and the ring of invariants is

$$A = \mathbb{C}[X_0, \dots, X_5]/(X_0 + \dots + X_5, X_0^3 + \dots + X_5^3).$$

Gel'fand MacPherson correspondence (geometric version):

$$M_{2\times n} = \left\{ \begin{bmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{bmatrix} \right\}$$

$$G(2,n) \cong M_{2\times n} /\!/ \mathrm{SL}(2,\mathbb{C}) \qquad\qquad M_{2\times n} /\!/ T \cong (\mathbb{P}^1)^n$$

$$G(2,n)/\!/ T \cong M_n \cong \mathbb{P}^1 /\!/ \mathrm{SL}(2,\mathbb{C})$$

- $G(2,n)$ is the Grassmannian of 2-planes in $\mathbb{C}^n$
- T is the torus

$$T = \left\{ \begin{bmatrix} t_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t_n \end{bmatrix} : t_1 \cdots t_n = 1 \right\}.$$

Gel'fand MacPherson correspondence (algebraic version):

$$\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$$

$$\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^{\mathrm{SL}(2,\mathbb{C})} \qquad \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^T$$

$$A \cong \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^{\mathrm{SL}(2,\mathbb{C}) \times T}$$

- $\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^T =$ polynomials that are multihomogeneous in x_i and y_i.
- $\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^{\mathrm{SL}(2,\mathbb{C})}$ is the homogeneous coordinate ring of $G(2, n)$ in the Plücker embedding.

Definition

Let

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array} = \det \begin{bmatrix} x_i & x_j \\ y_i & y_j \end{bmatrix} = x_i y_j - x_j y_i.$$

Theorem (First Fundamental Theorem of Invariant Theory)

The invariants $\begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array}$ *generate the invariant ring*

$$\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^{\mathrm{SL}(2,\mathbb{C})}.$$

These invariants satisfy the *Plücker relations*:

- $\begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array} = -\begin{array}{|c|}\hline j \\ \hline i \\ \hline \end{array}$
- $\begin{array}{|c|c|}\hline i & k \\ \hline j & l \\ \hline \end{array} = \begin{array}{|c|c|}\hline i & j \\ \hline k & l \\ \hline \end{array} + \begin{array}{|c|c|}\hline i & k \\ \hline l & j \\ \hline \end{array}.$

Definition

A Young tableau

i_1	$\cdots$	i_r
j_1	$\cdots$	j_r

is called *semistandard* if

- $i_t < j_t$ for all $1 \leqslant t \leqslant r$
- $i_1 \leqslant \cdots \leqslant i_r$
- $j_1 \leqslant \cdots \leqslant j_r$.

Theorem

The monomials corresponding to the $2 \times r$ *semistandard Young tableaux form a basis for the degree* r *part of the Plücker ring*

$$\left(\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^{\mathrm{SL}(2,\mathbb{C})}\right)_r .$$

Recall the Gel'fand MacPherson correspondence (algebraic version):

$$\mathbb{C}[x_1, y_1, \dots, x_n, y_n]$$

$$\mathbb{C}[x_1, y_1, \dots, x_n, y_n]^{\mathrm{SL}(2,\mathbb{C})} \qquad \mathbb{C}[x_1, y_1, \dots, x_n, y_n]^{T}$$

$$A \cong \mathbb{C}[x_1, y_1, \dots, x_n, y_n]^{\mathrm{SL}(2,\mathbb{C})\times T}$$

Definition

Let $\tau = \begin{array}{|c|c|c|}\hline i_1 & \cdots & i_r \\ \hline j_1 & \cdots & j_r \\ \hline \end{array}$ be a Young tableau, where $1 \leqslant i_\ell, j_\ell \leqslant n$.
Let

$$\mu_\ell = |\{k \mid i_k = \ell\} \cup \{k \mid j_k = \ell\}|.$$

The *filling* of τ is defined to be $\mu = (\mu_1, \ldots, \mu_n)$.

Example

If $\tau = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 3 & 4 \\ \hline \end{array}$, the filling is $\mu = (1, 1, 3, 1)$.

Let

$$t \in T = \left\{ \begin{bmatrix} t_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t_n \end{bmatrix} : t_1 \cdots t_n = 1 \right\}.$$

Then $t \cdot \begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array} = t_i x_i t_j y_j - t_j x_j t_i y_i = t_i t_j \begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array}$.

Let $\tau = \begin{array}{|c|c|c|}\hline i_1 & \cdots & i_r \\ \hline j_1 & \cdots & j_r \\ \hline \end{array}$ be a semistandard Young tableau with filling $(\mu_1, \dots, \mu_n)$.

Then

$$t \cdot \tau = t_{i_1} t_{j_1} \cdots t_{i_r} t_{j_r} \begin{array}{|c|c|c|}\hline i_1 & \cdots & i_r \\ \hline j_1 & \cdots & j_r \\ \hline \end{array} = t_1^{\mu_1} \cdots t_n^{\mu_n} \tau$$

is invariant under the torus action if and only if $(\mu_1, \dots, \mu_n) = (d, \dots, d)$.

Theorem (Kempe, 1894)

The ring of invariants A is generated by the lowest degree invariants.

- *When n is even, the SSYT of shape $2 \times \frac{n}{2}$ with filling $(1, \ldots, 1)$ generate A.*
- *When n is odd, the SSYT of shape $2 \times n$ with filling $(2, \ldots, 2)$ generate A.*

Example

- $n = 4$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}.$$

- $n = 5$:

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 3 & 4 & 4 & 5 & 5 \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 \\ \hline 2 & 3 & 3 & 5 & 5 \\ \hline \end{array}.$$

Fix n even (odd). Let $\mathbb{C}[X_\tau]$ be the polynomial ring in the variables X_τ, where τ runs over all SSYT of shape $2 \times \frac{n}{2}$ ($2 \times n$) and filling $(1, \ldots, 1)$ ($(2, \ldots, 2)$). Then by Kempe's theorem, we have

$$I \hookrightarrow \mathbb{C}[X_\tau] \twoheadrightarrow A,$$

where I is the ideal of relations between the generators of the ring of invariants.

Theorem (Howard, Millson, Snowden, Vakil, 2009)

When $n \neq 6$, then I is generated by equations of degree 2.

Does I admit a quadratic Gröbner basis?

Example

When $n = 8$, A is not Koszul. In particular there exists no term order $\prec$ such that $\mathrm{in}_{\prec} I$ is generated by quadratic monomials.

Theorem (Eisenbud, Reeves, Totaro)

Let

$$R = \mathbb{C}[x_1, \ldots, x_N]/J,$$

where J is a homogeneous ideal in R. Then for d large enough, the d'th Veronese subring

$$R^{(d)} = \bigoplus_m R_{md} \cong \mathbb{C}[x_1, \ldots, x_M]/J_d$$

where J_d has a quadratic Gröbner basis.

Theorem (-, Howard)

- *If n is even and k is even, then I_k has a quadratic Gröbner basis. In particular, I_2 has a quadratic Gröbner basis.*
- *If n is odd, then I admits a quadratic Gröbner basis.*

In both cases the initial ideal is square free.

Idea of Proof:

We assume n is odd.

Step 1: Degenerate the moduli space to a toric variety

Let $\tau = \begin{array}{|c|c|c|}\hline i_1 & \cdots & i_n \\ \hline j_1 & \cdots & j_n \\ \hline\end{array}$ be a SSYT of shape $2 \times n$ with filling $(2, \ldots, 2)$, and let

$$w_\tau\left(\begin{array}{|c|c|c|}\hline i_1 & \cdots & i_n \\ \hline j_1 & \cdots & j_n \\ \hline\end{array}\right) = \sum_{k=1}^{r} i_k + 2j_k.$$

Then $w = (w_\tau)$ is a weight vector on the polynomial ring $\mathbb{C}[X_\tau]$.

Theorem

The initial ideal $\mathrm{in}_w I$ *is a binomial ideal. The corresponding variety is a normal toric variety.*

(Sturmfels, Guinculea-Lakshmibai, Sturmfels-Speyer, Caldararu, Alexeeev-Brion, Foth-Hu, Howard-Millson-Snowden-Vakil).

So w lies in a face of the Gröbner fan, and is contained in $\mathrm{Trop}(M_n)$.

Theorem

The initial ideal $\mathrm{in}_w I$ *is a binomial ideal. The corresponding variety is a toric variety.*

(Hence there exists a *flat family* whose general fiber is isomorphic to M_n and whose special fiber is isomorphic to a toric variety, ref. Mutsihiro Miyazaki's talk).

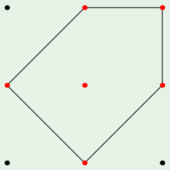
The corresponding polytope is given by

$$P = \{(a_1, \ldots, a_{n-3}) \in \mathbb{R}^{n-3} \mid 2 \geqslant a_1, a_{n-3} \geqslant 0, \\ a_i + a_{i+1} \geqslant 1, a_i + 1 \geqslant a_{i+1}, a_{i+1} + 1 \geqslant a_i\}$$

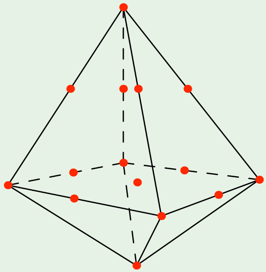
$$P = \{(a_1, \ldots, a_{n-3}) \in \mathbb{R}^{n-3} \mid 2 \geqslant a_1, a_{n-3} \geqslant 0, \\ a_i + a_{i+1} \geqslant 1, a_i + 1 \geqslant a_{i+1}, a_{i+1} + 1 \geqslant a_i\}$$

Example

The polytope for $n = 5$:
$2 \geqslant a_1, a_2, \geqslant 0,\ a_1 + a_2 \geqslant 1,$
$a_1 + 1 \geqslant a_2,\ a_2 + 1 \geqslant a_1.$



The polytope for $n = 6$



Step 2: We have

$$\mathbb{C}[X_\tau]/\operatorname{in}_w(I) \cong \mathbb{C}[X_u]/I_P,$$

where I_P is the toric ideal associated to the lattice polytpe P.
(The lattice points $u \in P \cap \mathbb{Z}^{n-3}$ are in one-to-one correspondence with the SSYT of shape $2 \times n$ with filling $(2, \ldots, 2)$.)
We define a term order $\prec$ on $\mathbb{C}[X_u]$.

- order the variables X_u for $u \in P \cap \mathbb{Z}^d$ using standard lexicographic ordering $\mathbb{Z}^d$
- Let $\prec_{\mathrm{dlex}}$ be the degree lexicographic order on $k[X_u]_{u \in P \cap \mathbb{Z}^d}$ induced by ordering of X_u.
- For a monomial $m = \prod_{i=1}^r X_{u_i}$, we define $N(m) = \sum_{i=1}^r \|u_i\|^2$.

We define $m_1 \prec m_2$ if

- $\deg(m_1) < \deg(m_2)$, or
- $\deg(m_1) = \deg(m_2)$ and $N(m_1) < N(m_2)$, or
- $\deg(m_1) = \deg(m_2)$, $N(m_1) = N(m_2)$, and $m_1 \prec_{\mathrm{dlex}} m_2$.

Theorem

The initial ideal $\mathrm{in}_{\prec} I_P$ *is generated by squarefree quadratic monomials.*

We get a term order $\prec_w$ on $\mathbb{C}[X_\tau]$, by letting $m_1 \prec_w m_2$ if

- $w(m_1) < w(m_2)$, or
- $w(m_1) = w(m_2)$ and $m_1 \prec m_2$.

Then

$$\mathrm{in}_{\prec_w} I = \mathrm{in}_{\prec}(\mathrm{in}_w I) = \mathrm{in}_{\prec}(I_P)$$

is generated by squarefree quadratic monomials.