Generating Differential Invariants and their Syzygies

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Algebra of Differential Invariants

$$\left(\mathbb{K}\,[\![y_1,\ldots,y_n]\!]\,/\,[\![s_1,\ldots,s_k]\!]\;,\;\{\mathcal{D}_1,\ldots,\mathcal{D}_m\}\right)$$

$$\mathcal{D}_i\mathcal{D}_j - \mathcal{D}_j\mathcal{D}_i = c_{ij1}\,\mathcal{D}_1 + \ldots + c_{ijm}\,\mathcal{D}_m, \qquad c_{ij}^k \in \mathbb{K}\,[\![y_1,\ldots,y_n]\!]$$

Motivation: symmetry reduction for differential elimination.

Collaborators: I. Kogan, **E. Mansfield**, G. Mari-Beffa, P. Olver.

Differential Elimination

$$\mathcal{S}\left\{\begin{array}{rcl} u\,(\phi_{xx}+\phi_{yy})+u_x\,\phi_x+u_y\,\phi_y+\phi & = & 0\\ u\,(\psi_{xx}+\psi_{yy})+u_x\,\psi_x+u_y\,\psi_y+\psi & = & 0\\ \psi_x\,\phi_x+\psi_y\,\phi_y & = & 0 \end{array}\right.$$

What are the conditions on u for $\mathcal{S}$ to have a solution?

Differential Polynomial Rings

$\mathbb{F}$ a field

$$\mathbb{F} = \mathbb{Q}(x, y)$$

$\mathrm{D}_1, \ldots, \mathrm{D}_m$ derivations on $\mathbb{F}$

$$\mathrm{D}_1 = \frac{\partial}{\partial x},\ \mathrm{D}_2 = \frac{\partial}{\partial y}$$

$$\mathcal{Y} = \{y_1, \ldots, y_n\}$$

$$\mathcal{Y} = \{\phi, \psi\}$$

$$\mathbb{F}[\, y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}\,] = \mathbb{F}\,[\![\mathcal{Y}]\!]$$

$$\mathbb{F}\,[\![\phi, \psi]\!] = \mathbb{F}[\phi, \phi_x, \phi_y, \ldots, \psi \ldots]$$

$$\phi_{xxy} \rightsquigarrow \phi_{x^2y} \rightsquigarrow \phi_{(2,1)}$$

$$\mathrm{D}_i\,(y_\alpha) = y_{\alpha+\epsilon_i}$$

$$\frac{\partial}{\partial x}\,(\phi_{xxy}) = \phi_{xxxy} \rightsquigarrow \mathrm{D}_1\,\left(\phi_{(2,1)}\right) = \phi_{(3,1)}$$

$$\epsilon_i = (0, \ldots, \underset{i^{\text{th}}}{1}, \ldots, 0)$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$$

$$\mathrm{D}_i\mathrm{D}_j = \mathrm{D}_j\mathrm{D}_i$$

Derivations with nontrivial commutations

$$\mathcal{Y} = \{y_1, \ldots, y_n\}$$

$$\mathcal{D} = \{\mathcal{D}_1, \ldots, \mathcal{D}_m\}$$

$$\mathcal{D}_i \, \mathcal{D}_j - \mathcal{D}_j \, \mathcal{D}_i = \sum_{l=1}^{m} c_{ijl} \, \mathcal{D}_l$$

$$c_{ijl} \in \mathbb{K} \, [\![\mathcal{Y}]\!]$$

$$\mathbb{K} \, [\![\mathcal{Y}]\!]?$$

Differential polynomial ring $\mathbb{K}[\![\mathcal{Y}]\!]$ with non commuting derivations

$$\mathcal{Y} = \{y_1, \ldots, y_n\}$$
$$\mathrm{D} = \{\mathcal{D}_1, \ldots, \mathcal{D}_m\}$$
$$\mathbb{K}[y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}]$$

$$\mathcal{D}_i(y_\alpha) = \begin{cases} y_{\alpha+\epsilon_i} \\ \quad \text{if } \alpha_1 = \ldots = \alpha_{i-1} = 0 \\ \mathcal{D}_j\mathcal{D}_i\left(y_{\alpha-\epsilon_j}\right) + \sum_{l=1}^{m} c_{ijl}\,\mathcal{D}_l\left(y_{\alpha-\epsilon_j}\right) \\ \quad \text{where } j < i \text{ is s.t. } \alpha_j > 0 \\ \quad \text{while } \alpha_1 = \ldots = \alpha_{j-1} = 0 \end{cases}$$

If the c_{ijl} satisfy

- $c_{ijl} = -c_{jil}$
- $\mathcal{D}_k(c_{ijl}) + \mathcal{D}_i(c_{jkl}) + \mathcal{D}_j(c_{kil}) = \sum_{\mu=1}^{m} c_{ij\mu}c_{\mu kl} + c_{jk\mu}c_{\mu il} + c_{ki\mu}c_{\mu jl}$

& there exists an *admissible ranking* $\prec$

- $|\alpha| < |\beta| \ \Rightarrow\ y_\alpha \prec y_\beta$,
- $y_\alpha \prec z_\beta \ \Rightarrow\ y_{\alpha+\gamma} \prec z_{\beta+\gamma}$,
- $\sum_{l\in\mathbb{N}_m} c_{ijl}\,\mathcal{D}_l(y_\alpha) \prec y_{\alpha+\epsilon_i+\epsilon_j}$

then

$$\mathcal{D}_i\mathcal{D}_j(p) - \mathcal{D}_j\mathcal{D}_i(p) = \sum_{l=1}^{m} c_{ijl}\,\mathcal{D}_l(p) \quad \forall p \in \mathbb{K}[\,y_\alpha \mid \alpha \in \mathbb{N}^m\,]$$

[H05]

Generating Differential Invariants and their Syzygies

1. Invariants of Lie group actions
2. Normalized Invariants: Geometric Construction
3. Differential Invariants, Invariant Derivations
4. Algebra of Differential Invariants

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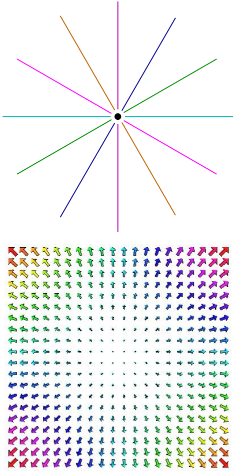
One dimensional Lie group actions on the plane

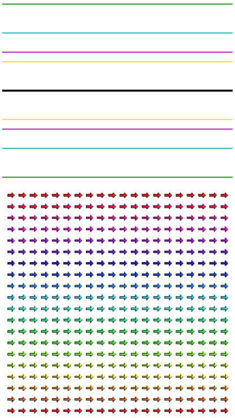
	scaling	translation	rotation
Group	$\mathbb{R}^*$	$\mathbb{R}$	$SO(2)$
$\lambda \star \begin{pmatrix} x \\ y \end{pmatrix}$	$\begin{pmatrix} \lambda\, x \\ \lambda\, y \end{pmatrix}$	$\begin{pmatrix} x+\lambda \\ y \end{pmatrix}$	$\begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Orbits:

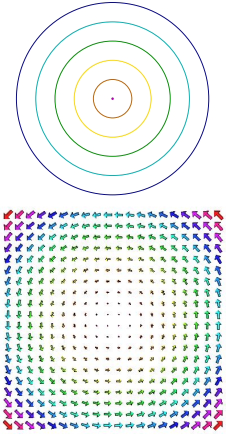
Infinitesimal generator



$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$



$$\frac{\partial}{\partial x}$$



$$y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

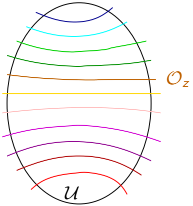
Infinitesimal generators

$$\xi_1 \frac{\partial}{\partial z_1} + \ldots + \xi_d \frac{\partial}{\partial z_d}$$

a vector field the flow of which is the action of a one-dimensional group.

$\mathrm{V}_1, \ldots, \mathrm{V}_r$ a basis of infinitesimal generators for the action of r-dimensional group $\mathcal{G}$.

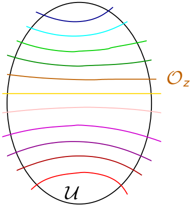
Invariants



$f : \mathcal{M} \to \mathbb{R}$ smooth

$$f(\lambda \star z) = f(z) \text{ for } \lambda \in \mathcal{G}$$
$$\Leftrightarrow$$
$$f \text{ is constant on orbits}$$

Local Invariants



$f : \mathcal{U} \subset \mathcal{M} \to \mathbb{R}$ smooth

$f(\lambda \star z) = f(z)$ for $\lambda \in \mathcal{G}$ close to e

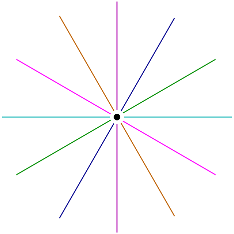
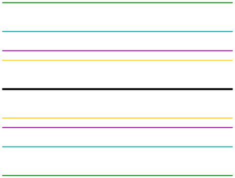
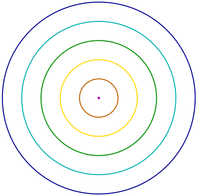
$\Leftrightarrow$

f is constant on orbits within $\mathcal{U}$

$\Leftrightarrow$

$$\mathrm{V}_1(f) = 0, \ldots, \mathrm{V}_r(f) = 0$$

Examples

$\mathcal{G}$	$\mathbb{R}^*$	$\mathbb{R}$	$SO(2)$
			
V	$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$\frac{\partial}{\partial x}$	$y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$
Invariant	$\frac{x}{y}$	y	$\sqrt{x^2 + y^2}$

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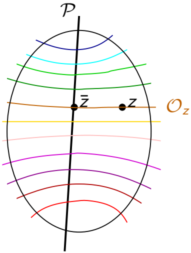
Local cross-section $\mathcal{P}$

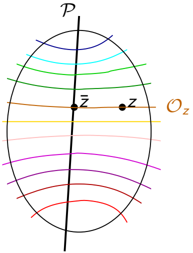


- $\mathcal{P}$ an embedded manifold of dimension $n - d$

- $\mathcal{P}$ intersect $\mathcal{O}_z^0$ at a unique point, $\forall z \in \mathcal{U}$.
- $\mathcal{P}$ is transverse to $\mathcal{O}_z$ at $z \in \mathcal{P}$.

[Fels Olver 99, H. Kogan 07b]

Local cross-section $\mathcal{P}$



- $\mathcal{P}$ an embedded manifold of dimension $n-d$

  $\mathcal{P} = \{z \in \mathcal{U} \mid p_1(z) = \ldots = p_d(z) = 0\}$

- $\mathcal{P}$ intersect $\mathcal{O}_z^0$ at a unique point, $\forall z \in \mathcal{U}$.
- $\mathcal{P}$ is transverse to $\mathcal{O}_z$ at $z \in \mathcal{P}$.

  $\Leftrightarrow \; V(P) = (V_i(p_j))_{ij}$ has rank d on $\mathcal{P}$.

[Fels Olver 99, H. Kogan 07b]

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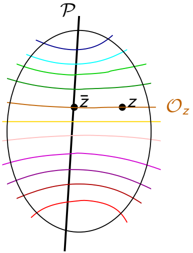

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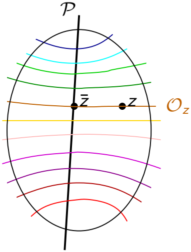
- $\mathcal{P}$ intersect $\mathcal{O}_z^0$ at a unique point, $\forall z \in \mathcal{U}$.
- $\mathcal{P}$ is transverse to $\mathcal{O}_z$ at $z \in \mathcal{P}$.

A local invariant is uniquely determined by a function on $\mathcal{P}$.

[Fels Olver 99, H. Kogan 07b]

Invariantization $\bar{\iota}f$ of a function f



$f : \mathcal{U} \to \mathbb{R}$ smooth

$\bar{\iota}f$ is the unique *local* invariant with $\bar{\iota}f|_{\mathcal{P}} = f|_{\mathcal{P}}$

$$\bar{\iota}f(z) = f(\bar{z})$$

Normalized invariants: $\bar{\iota}z_1, \ldots, \bar{\iota}z_n$. $\qquad \bar{\iota}f(z) = f(\bar{\iota}z)$

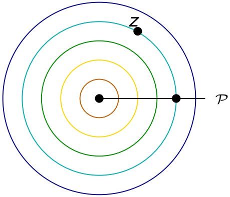
Generation and rewriting:

$$f \text{ local invariant } \Rightarrow f(z_1, \ldots, z_n) = f(\bar{\iota}z_1, \ldots, \bar{\iota}z_n)$$

Relations: $p_1(\bar{\iota}z_1, \ldots, \bar{\iota}z_n) = 0, \ldots, p_d(\bar{\iota}z_1, \ldots, \bar{\iota}z_n) = 0$

[Fels Olver 99, H. Kogan 07b]

Normalized invariants. Example.



$\mathcal{G} = SO(2)$, $\qquad \mathcal{M} = \mathbb{R}^2 \setminus O$

$\mathcal{P} : y = 0,\ x > 0 \qquad \mathcal{U} = \mathcal{M}$

$$(\bar{\iota}x, \bar{\iota}y) = \left(\sqrt{x^2 + y^2}, 0\right)$$

Replacement property:

$$f(x, y) \text{ invariant } \Rightarrow f(x, y) = f\left(\bar{\iota}x, 0\right).$$

Computing normalized invariants

In the algebraic case, the normalized invariants $(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$ form a $\overline{\mathbb{K}(z)}^G$-zero of the *graph-section* ideal

$$(G + (Z - \lambda \star z) + P) \cap \mathbb{K}(z)[Z]$$

The coefficients of the reduced Gröbner basis of the graph-section ideal form a generating set for $\mathbb{K}(z)^G$ endowed with a simple rewriting algorithm.

[H. Kogan 07a 07b]

Normalized invariants in practice

We mostly do not need $(\bar{\iota}z_1, \ldots, \bar{\iota}z_n)$ explicitly.

We can work formally with $(\bar{\iota}z_1, \ldots, \bar{\iota}z_n)$, subject to the relationships $p_1(\bar{\iota}z) = 0, \ldots, p_d(\bar{\iota}z) = 0$.

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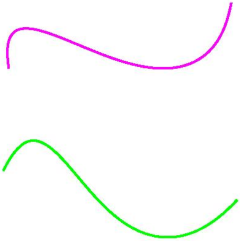
Classical differential invariants



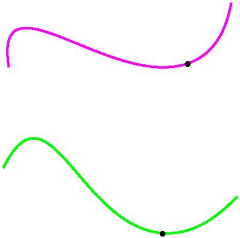
$E(2)$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \xi & -\zeta \\ \zeta & \xi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\xi^2 + \zeta^2 = 1$$

Curvature: $\sigma = \sqrt{\frac{y_{xx}^2}{(1+y_x^2)^3}}$

Classical differential invariants



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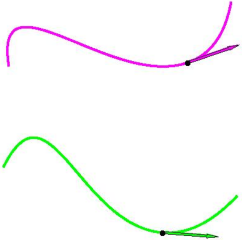

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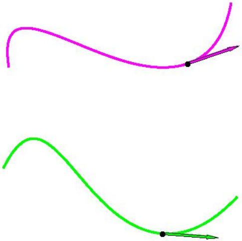
$$\xi^2 + \zeta^2 = 1$$

$$Y_X = \frac{\zeta + \xi y_x}{\xi - \zeta\, y} \qquad Y_{XX} = \frac{y_{xx}}{(\xi - \zeta\, y)^3}$$

Curvature: $\sigma = \sqrt{\frac{y_{xx}^2}{(1+y_x^2)^3}}$ a differential invariant

Classical differential invariants



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Invariant derivation: $\dfrac{d}{ds} = \dfrac{1}{\sqrt{1+y_x^2}} \dfrac{d}{dx}$

Jets and Action Prolongation

$\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$

$(x_1, \ldots, x_m)$ coordinates on $\mathcal{X} \rightsquigarrow$ independent variables

$(u_1, \ldots, u_n)$ coordinates on $\mathcal{U} \rightsquigarrow$ dependent variables

$\mathrm{J}^k = \mathcal{X} \times \mathcal{U}^{(k)}$

additional coordinates $u_\alpha = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}$, $|\alpha| \leq k$

$\rightsquigarrow$ the derivatives of u w.r.t x up to order k

$$\mathrm{D}_i = \frac{\partial}{\partial x_i} + \sum_\alpha u_{\alpha+\epsilon_i} \frac{\partial}{\partial u_\alpha}$$

Jets and Action Prolongation

$\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ $\qquad$ $g^{(0)} : \mathcal{G} \times \mathrm{J}^0 \to \mathrm{J}^0 \quad \mathrm{V}_1^0, \ldots, \mathrm{V}_r^0$

$\mathrm{J}^k = \mathcal{X} \times \mathcal{U}^{(k)}$ $\qquad$ $g^{(k)} : \mathcal{G} \times \mathrm{J}^k \to \mathrm{J}^k \quad \mathrm{V}_1^k, \ldots, \mathrm{V}_r^k$

[DifferentialGeometry]

Differential Invariants

$\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ $\qquad g^{(0)} : \mathcal{G} \times \mathrm{J}^0 \to \mathrm{J}^0 \qquad \mathrm{V}^0_1, \ldots, \mathrm{V}^0_r$

$\mathrm{J}^k = \mathcal{X} \times \mathcal{U}^{(k)}$ $\qquad g^{(k)} : \mathcal{G} \times \mathrm{J}^k \to \mathrm{J}^k \qquad \mathrm{V}^k_1, \ldots, \mathrm{V}^k_r$

$f : \mathrm{J}^k \to \mathbb{R}$ differential invariant of order k if $\mathrm{V}^k(f) = 0$.

Given a cross-section $\mathcal{P}^k$ on J^k, we define the normalized invariants of order k

$$\mathcal{I}^k = \{\bar{\iota} x_1, \ldots, \bar{\iota} x_m\} \cup \{\bar{\iota} u_\alpha \,|\, |\alpha| \leq k\}$$

Invariant derivation

$$\mathcal{D} : \mathcal{F}(\mathrm{J}^k) \to \mathcal{F}(\mathrm{J}^{k+1}) \;\; \text{s.t} \;\; \mathcal{D} \circ \mathrm{V} = \mathrm{V} \circ \mathcal{D}$$

$f : \mathrm{J}^k \to \mathbb{R}$ a differential invariant

$\Rightarrow \;\; \mathcal{D}(f)$ a differential invariant of order $k+1$.

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Moving frame

The dimension of orbits on J^k, r_k, stabilizes:

$$r_0 \leq r_1 \leq \ldots \leq r_s = r_{s+1} = \ldots = r.$$

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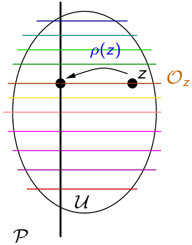
$\mathcal{P}^s : p_1 = 0, \ldots, p_r = 0$ a cross-section on J^{s+k}



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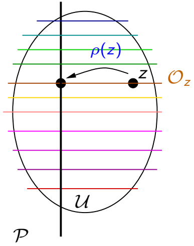
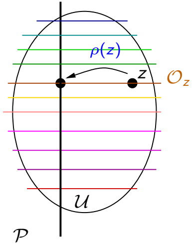


$$\rho : \mathrm{J}^k \to \mathcal{G} \quad \text{equivariant} \quad \rho(\lambda \star z) = \rho(z) \cdot \lambda^{-1}$$

Moving frame



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$\rho : \mathrm{J}^k \to \mathcal{G}$ equivariant $\rho(\lambda \star z) = \rho(z) \cdot \lambda^{-1}$

Invariant derivations:

$$\begin{pmatrix} \mathcal{D}_1 \\ \vdots \\ \mathcal{D}_m \end{pmatrix} = \rho^* \left(\mathrm{D}_i(\lambda \bullet x_j)\right)_{ij} \begin{pmatrix} \mathrm{D}_1 \\ \vdots \\ \mathrm{D}_m \end{pmatrix}$$

[Fels Olver 99]

Invariant Derivations

$$\mathcal{D}_1, \ldots, \mathcal{D}_m : \mathcal{F}(\mathrm{J}^{s+k}) \to \mathcal{F}(\mathrm{J}^{s+k+1})$$

$$\mathcal{D}_i(\bar{\iota} f) = \bar{\iota}(\mathrm{D}_i(f)) - K_{ia}\, \bar{\iota}(\mathrm{V}_a(f)) \qquad K = \bar{\iota}\left(\mathrm{D}(P)\, \mathrm{V}(P)^{-1}\right)$$

$$D(P) = (\mathrm{D}_i(p_j)) \qquad \mathrm{V}(P) = (\mathrm{V}_i(p_j))$$

[Fels Olver 99]

Invariant Derivations

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$$\mathcal{D}_i(\,\bar{\iota} f\,) = \bar{\iota}\,(\,\mathrm{D}_i(f)\,) - K_{ia}\,\bar{\iota}\,(\mathrm{V}_a(f)) \qquad K = \bar{\iota}\,\big(\mathrm{D}(P)\,\mathrm{V}(P)^{-1}\big)$$

$$\big[\mathcal{D}_i, \mathcal{D}_j\big] = \sum_{k=1}^{m} \Lambda_{ijk}\,\mathcal{D}_k \qquad \Lambda_{ijk} = K_{ic}\bar{\iota}(\mathrm{D}_j(\xi_{ck})) - K_{jc}\bar{\iota}(\mathrm{D}_i(\xi_{ck}))$$

[Fels Olver 99]

Finite Generation

$\bar{\iota} u_{\alpha+\epsilon_i} = \mathcal{D}_i(\bar{\iota} u_\alpha) + K_{ia}\, \bar{\iota}\,(\mathrm{V}_a(u_\alpha))$ $\qquad K = \bar{\iota}\left(\mathrm{D}(P)\,\mathrm{V}(P)^{-1}\right)$

Any differential invariant can be contructively written in terms of either:

- the normalized invariants of order $s+1$

$$\mathcal{I}^{s+1} = \{\bar{\iota} x_1, \ldots, \bar{\iota} x_m\} \cup \{\bar{\iota} u_\alpha \,|\, |\alpha| \leq s+k\}$$

- the *edge invariants*, when the cross-section is of minimal order

$$\mathcal{E} = \{\bar{\iota}(\mathcal{D}_i(p_a))\} \cup \mathcal{I}^0,$$

- the *Maurer-Cartan invariants* $\qquad \mathcal{K} = \{K_{ia}\} \cup \mathcal{I}^0$

and their derivatives w.r.t. $\mathcal{D}_1, \ldots, \mathcal{D}_m$

[H09, H10]

Syzygies for Normalized Invariants

A subset S of the following relationships

$$p_1(\bar{\iota}x, \bar{\iota}u_\alpha) = 0, \ \ldots, \ p_r(\bar{\iota}x, \bar{\iota}u_\alpha) = 0$$

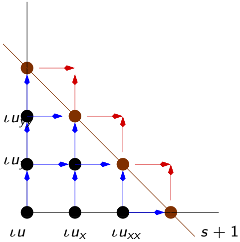
$$\mathcal{D}_i(\bar{\iota}x_j) = \delta_{ij} - K_{ia}\bar{\iota}(\mathrm{V}(x_j)),$$

$$\mathcal{D}_i(\bar{\iota}u_\alpha) = \bar{\iota}u_{\alpha+\epsilon_i} - K_{ia}\bar{\iota}(\mathrm{V}(u_\alpha)), \ |\alpha| \le s$$

$$\mathcal{D}_i(\bar{\iota}u_\alpha) - \mathcal{D}_j(\bar{\iota}u_\beta) = K_{ja}\bar{\iota}(\mathrm{V}(u_\beta)) - K_{ia}\bar{\iota}(\mathrm{V}(u_\alpha)),$$

$$\alpha + \epsilon_i = \beta + \epsilon_j, \ |\alpha| = |\beta| = s + 1.$$

form a *complete set of differential syzygies*.



$$\bar{\iota}u_{\alpha+\epsilon_i} = \mathcal{D}_i(\bar{\iota}u_\alpha) + K_{ia}\,\bar{\iota}(\mathrm{V}_a(u_\alpha))$$

[H09]

Syzygies for Maurer-Cartan Invariants

$$\mathcal{D}_i(K_{jc}) - \mathcal{D}_j(K_{ic}) = \sum_{1\le a<b\le r} C_{abc}(K_{ia}K_{jb} - K_{ja}K_{ib}) + \sum_{k=1}^{m} \Lambda_{ijk}\, K_{kc} = 0$$

where

$$[\mathrm{v}_i, \mathrm{v}_j] = \sum_{k=1}^{r} C_{ijk}\mathrm{v}_k \qquad \left[\mathcal{D}_i, \mathcal{D}_j\right] = \sum_{k=1}^{m} \Lambda_{ijk}\, \mathcal{D}_k .$$

ありがとう.

Thanks.

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