# Gröbner bases over fields with valuations

**Diane Maclagan**

Osaka, June 2010

## Valuations

$S = K[x_0, \ldots, x_n]$, where $K$ is a field with a **valuation** $\text{val} : K \to \mathbb{R} \cup \{\infty\}$ satisfying:

$\text{val}(ab) = \text{val}(a) + \text{val}(b)$,
$\text{val}(a + b) \geq \min(\text{val}(a), \text{val}(b))$, and
$\text{val}(a) = \infty$ if and only if $a = 0$.

**Examples:** $K = \mathbb{Q}(t)$. $\text{val}(f/g) = \text{lowdeg}(f) - \text{lowdeg}(g)$.
$\text{val}((3t^2 + 7t^3 + 8t^6)/(5t^4 - 7t^5)) = -2$.

$K = \mathbb{Q}$. $\text{val}(p^n a/b) = n$, where $p \not| a, b$. **p-adic valuation**.

For $p = 2$, $\text{val}(4) = 2$, $\text{val}(7) = 0$, and $\text{val}(5/6) = -1$.

## Valuations continued

The ring $R = \{a \in K : \mathrm{val}(a) \geq 0\}$ is a local ring with maximal ideal $\mathfrak{m} = \{a \in K : \mathrm{val}(a) > 0\}$. The quotient $\Bbbk = R/\mathfrak{m}$ is the **residue field**.

**Examples:** For $K = \mathbb{Q}(t)$, the ring $R$ is $\mathbb{Q}[t]_{\langle t \rangle}$, $\mathfrak{m} = \langle t \rangle$, and $\Bbbk = R/\mathfrak{m} \cong \mathbb{Q}$.

For $K = \mathbb{Q}$ with the $p$-adic valuation, $R$ is $\mathbb{Z}_{\langle p \rangle}$, $\mathfrak{m} = \langle p \rangle$, and $\Bbbk = R/\mathfrak{m} \cong \mathbb{Z}/p\mathbb{Z}$.

Choose a homomorphism $\Gamma = \mathrm{im}\,\mathrm{val} \to K$, $w \mapsto t^w$, with $\mathrm{val}(t^w) = w$.

**Examples:** $K = \mathbb{Q}(t)$, $t^w = t^w$.

$K = \mathbb{Q}$ with the $p$-adic valuation, $t^w = p^w$.

## Usual Gröbner bases

Fix $w \in \mathbb{R}^{n+1}$. For a polynomial $f = \sum c_u x^u$, let $W = \max\{w \cdot u\}$.

The **initial form** of $f$ is $\text{in}_w(f) = \sum_{w \cdot u = W} c_u x^u \in K[x_0, \ldots, x_n]$.

**Example:** $f = \langle 7x + 8y + 12z \rangle \subset \mathbb{Q}[x, y, z]$, $w = (3, 2, 1)$.
Then $\text{in}_w(f) = 7x$.

The **initial ideal** of $I \subset K[x_0, \ldots, x_n]$ is $\text{in}_w(I) = \langle \text{in}_w(f) : f \in I \rangle$.

## Gröbner bases with valuations

Fix $w \in \Gamma^{n+1}$. For a polynomial $f = \sum c_u x^u$, let
$W = \min\{\mathrm{val}(c_u) + w \cdot u\}$.

The **initial form** of $f$ is $\mathrm{in}_w(f) = \sum_{\mathrm{val}(c_u)+w\cdot u=W} \overline{t^{-\mathrm{val}(c_u)} c_u} x^u \in \Bbbk[x_0, \ldots, x_n]$.

**Example:** $f = \langle 7x + 8y + 12z \rangle \subset \mathbb{Q}[x, y, z]$, $w = (3, 2, 1)$. With the 2-adic valuation,
$\mathrm{in}_w(f) = x + z \in \mathbb{Z}/2\mathbb{Z}[x, y, z]$.

The **initial ideal** of $I \subset K[x_0, \ldots, x_n]$ is $\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle$.

**Motivation:** **Tropical Geometry.**

**Definition:** Let $I \subset K[x_0, \ldots, x_n]$. The **tropical variety**, $\mathrm{trop}(V(I))$ of $I$ is the closure in $\mathbb{R}^{n+1}$ of

$$\{w \in \Gamma_{\mathbb{Q}}^n : \mathrm{in}_w(I) \text{ does not contain a monomial } \}.$$

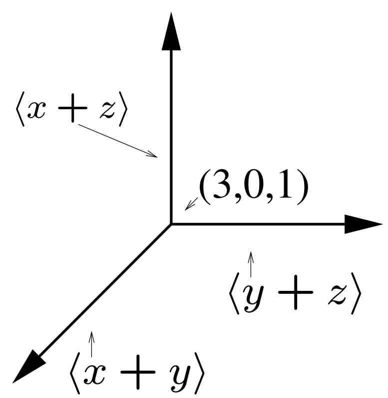
**Note:** If $w \in \mathrm{trop}(V(I))$ then $w + \lambda(1, 1, \ldots, 1)$ for all $\lambda \in \Gamma_{\mathbb{Q}}$. We thus consider $\mathrm{trop}(V(I)) \in \mathbb{R}^{n+1}/\mathbb{R}(1, \ldots, 1) \cong \mathbb{R}^n$.

**Motivation: Tropical Geometry.**

**Definition:** Let $I \subset K[x_0, \ldots, x_n]$. The **tropical variety**, $\mathtt{trop}(V(I))$ of $I$ is the closure in $\mathbb{R}^{n+1}$ of

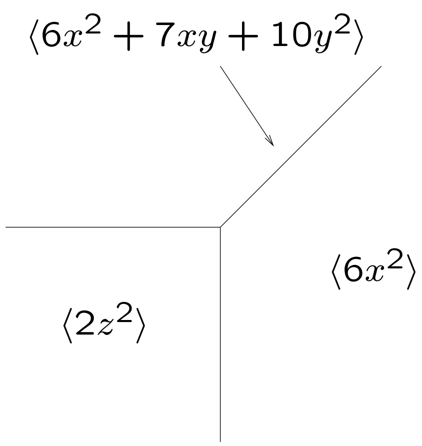
$$\{w \in \Gamma_{\mathbb{Q}}^n : \text{in}_w(I) \text{ does not contain a monomial } \}.$$

**Example:** $I = \langle 7x + 8y + 12z \rangle$.



**Usual Gröbner bases** Fix a homogeneous ideal $I \subset K[x_0, \ldots, x_n]$. Set $w \sim w'$ if $\mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$. The closures of equivalence classes are rational polyhedral cones, which together make up the **Gröbner fan** of $I$.

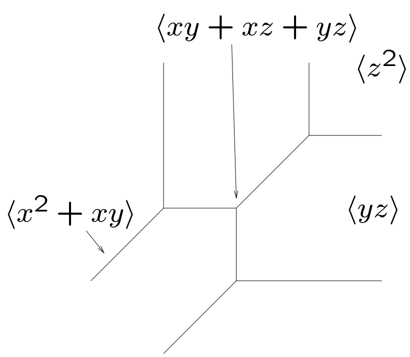
**Example:** Let $I = \langle 6x^2 + 7xy + 10y^2 + 5xz - yz + 2z^2 \rangle$.



**Gröbner bases with valuations.**

Fix a homogeneous ideal $I \subset K[x_0, \ldots, x_n]$. Set $w \sim w'$ if $\mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$. The closures of equivalence classes are $\Gamma$-rational polyhedra, which together make up the **Gröbner complex** of $I$.

**Example:** Let $I = \langle 6x^2 + 7xy + 10y^2 + 5xz - yz + 2z^2 \rangle$.



**Question:** How can we compute $\mathrm{in}_w(I)$?

**Question:** How can we compute $\text{in}_w(I)$?

**Answer 1:** For $K = \mathbb{Q}(t)$, we can clear denominators in the coefficients, and do the computation in $\mathbb{Q}[t, x_0, \ldots, x_1]$. Then

$$\text{in}_{(1,-w)}(I)|_{t=1} = \text{in}_w(I).$$

**Problem:** Doesn't work for the $p$-adics.

**Question:** How can we compute $\mathrm{in}_w(I)$?

**Answer 2:** (Usual Gröbner bases)

Buchberger algorithm:

Compute the normal form with respect to the generating set of all $S$-pairs of pairs of generators. Add to the generating set if it is nonzero, and repeat until the generating set stabilizes.

**Question:** How can we compute $\mathrm{in}_w(I)$?

**Answer 2:** (Usual Gröbner bases)

Buchberger algorithm:

Compute the normal form with respect to the generating set of all $S$-pairs of pairs of generators. Add to the generating set if it is nonzero, and repeat until the generating set stabilizes.

**Example:**

Let $I = \langle \underline{x} - 2y, \underline{y} - 2z, \underline{z} - 2x \rangle$, $w = (1, 1, 1)$, and $f = x$.

Then when we reduce $f$ by the generators for $I$ we get:

**Example:**

Let $I = \langle \underline{x} - 2y, \underline{y} - 2z, \underline{z} - 2x \rangle$, $w = (1, 1, 1)$, and $f = x$.

Then when we reduce $f$ by the generators for $I$ we get:

$$x \rightsquigarrow 2y$$

**Example:**

Let $I = \langle \underline{x} - 2y, \underline{y} - 2z, \underline{z} - 2x \rangle$, $w = (1, 1, 1)$, and $f = x$.

Then when we reduce $f$ by the generators for $I$ we get:

$$\begin{aligned} x &\rightsquigarrow 2y \\ &\rightsquigarrow 4z \end{aligned}$$

**Example:**

Let $I = \langle \underline{x} - 2y, \underline{y} - 2z, \underline{z} - 2x \rangle$, $w = (1, 1, 1)$, and $f = x$.

Then when we reduce $f$ by the generators for $I$ we get:

$$\begin{aligned} x &\rightsquigarrow 2y \\ &\rightsquigarrow 4z \\ &\rightsquigarrow 8x \end{aligned}$$

**Example:**

Let $I = \langle \underline{x} - 2y, \underline{y} - 2z, \underline{z} - 2x \rangle$, $w = (1, 1, 1)$, and $f = x$.

Then when we reduce $f$ by the generators for $I$ we get:

$$\begin{aligned} x &\rightsquigarrow 2y \\ &\rightsquigarrow 4z \\ &\rightsquigarrow 8x \\ &\rightsquigarrow \ldots \end{aligned}$$

**Solution:** Use a modified Mora standard basis algorithm.

This is (work-in-progress) implemented for the $p$-adics in a `Macaulay 2` package with Andrew Chan.

**Issues:** The valuation ring $R$ need not be Noetherian (eg if $K$ is the Puiseux series).

Requires "generic" weight vector $w$.

```
Macaulay2, version 1.3.1
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, SchurRings, TangentCone

i1 : installPackage ``GrobnerValuations''
--installing package GrobnerValuations in ../Library/Application Support/M

...

i2 : R= QQ[x,y,z,w];

i3 : I={4*x^3+x^2*y+32*z^3+4*y^3, 16*x^3+32*x*y^2+8*y^3+z*w^2};

i4 : grobnerVal({1,11/10,3,1},2,I)

         3    2      3      3     3          2     3       2
o4 = {4x  + x y + 4y  + 32z , 16x  + 32x*y  + 8y  + z*w }
```

```
i3 : I={4*x^3+x^2*y+32*z^3+4*y^3, 16*x^3+32*x*y^2+8*y^3+z*w^2};
...

i5 : grobnerVal({1,11/10,3,1},199,I)

        3       2   1 3    1   2   2         2     3      3   1   2
o5 = {x  + 2x*y  + -y  + --z*w , x y - 8x*y  + 2y  + 32z  - -z*w ,
                    2    16                                 4
     ----------------------------------------------------------------
        3    31 4   1   3      3    1      2    33      2   5
     x*y  - ---y  - -x*z  - 4y*z  + ---x*z*w  + ----y*z*w , y  +
            128     2               256         1024
     ----------------------------------------------------------------
     8192 2 3   1984     3   15872 2 3    64  2   2    63        2
     ----x z  + ----x*y*z  + -----y z  - ----x z*w  - ----x*y*z*w  +
     1985       1985            1985      1985          3970
     ----------------------------------------------------------------
       1   2   2
     -----y z*w }
     15880
```

## Remaining issues

1. What is the most efficient choice for the écart function of Mora's algorithm?

2. How can we determine if a given $w$ is sufficiently generic?

3. What is the complexity of this algorithm?

4. We need better implementations! ($p$-adic `gfan`!)