Gröbner bases in tropical geometry

Anders Jensen

Courant Research Center, Mathematisches Institut,
Georg-August-Universität Göttingen

Osaka, July 2010

Gröbner bases in tropical geometry

...or tropical geometry in Gröbner basis theory.

Outline:

- Gröbner fans
- Tropical varieties
- Properties and computational approaches
- Dimension arguments with an example

An algorithmic definition of Gröbner fans

Buchberger's algorithm:

- Input 1: A list of generators for an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$
- Input 2: A term order $\prec$ (represented by a vector in $\mathbb{R}^n_{\geq 0}$)
- Output: A reduced Gröbner basis for I w.r.t. $\prec$

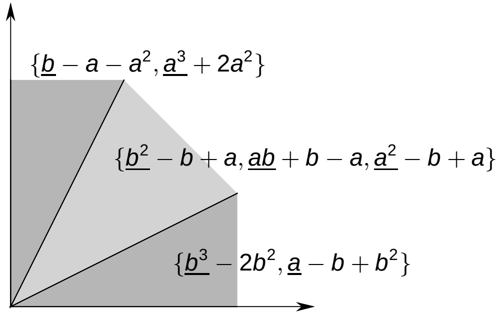
Observe:

- Varying Input 2 we get different Gröbner bases.
- Two vectors are *equivalent* if they produce the same Gröbner basis.
- The equivalence classes are the maximal cones in the *Gröbner fan of I*.

An algorithmic definition of Gröbner fans

Example

$I = \langle a - b - ab, a^2 + ab \rangle$



Initial forms and initial ideals

Consider the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Let $\omega \in \mathbb{R}^n$.

- The ω-*degree* of a monomial $x_1^{a_1} \cdots x_n^{a_n}$ with $a \in \mathbb{N}^n$ is $\langle \omega, a \rangle$.
- The *initial form* $in_\omega(f)$ of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is the sum of terms with maximal ω-*degree*.
  Example:

$$in_{(1,2)}(x_1^4 + 2x_2^2 + x_1 x_2 + 1) = x_1^4 + 2x_2^2$$

- The *initial ideal* of an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is defined as

$$in_\omega(I) = \langle in_\omega(f) : f \in I \rangle$$

The Gröbner fan of an ideal

Definition (Mora, Robbiano, 1988)

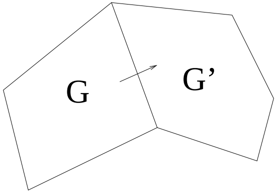
- Let $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ be a homogeneous ideal.
- Define the Gröbner cone at ω:

$$C_\omega(I) := \overline{\{u \in \mathbb{R}^n : \mathrm{in}_u(I) = \mathrm{in}_\omega(I)\}}.$$

- The set $\{C_\omega(I) : \omega \in \mathbb{R}^n\}$ is the *Gröbner fan* of I.

Algorithm (Collart, Kalkbrener, Mall, 1997)

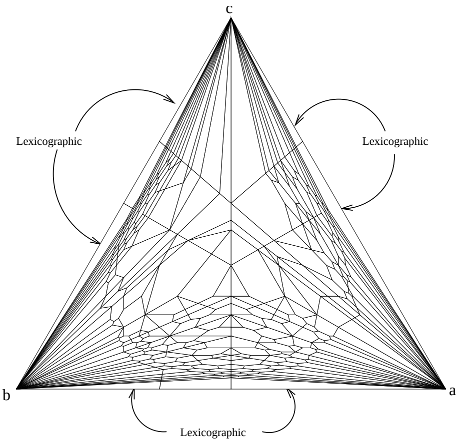
Gröbner "walk":



A bigger Gröbner fan example

Example

$I = \langle a^5 + b^3 + c^2 - 1, a^2 + b^2 + c - 1, a^6 + b^5 + c^3 - 1 \rangle \subseteq \mathbb{C}[a, b, c]$
has 360 reduced Gröbner bases and 360 full-dimensional cones in its fan. (Not homogeneous!) Intersection with triangle:



Tropical varieties

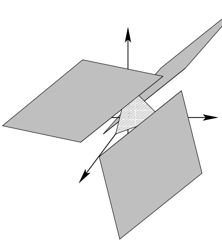
Definition

If $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is an ideal then we define

$$T(I) := \{\omega \in \mathbb{R}^n : \mathrm{in}_\omega(I) \text{ is monomial-free}\}.$$

Example

The tropical variety of a principal ideal is called a *tropical hypersurface*. $T(\langle x_1 + x_2 + x_3 \rangle) \subseteq \mathbb{R}^3$ is the union of three 2-dimensional cones:



Lemma

Any tropical variety is an intersection of hypersurfaces:

$$T(I) = \bigcap_{f \in I} T(\langle f \rangle)$$

A naive algorithm for computing the tropical variety

Algorithm

Input *Generators for homogeneous* $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$.

Output *The set of Gröbner cones contained in* $T(I)$.

- *Compute the Gröbner fan*
- *For each face C:*
  - *Compute a relative interior* $\omega \in C$
  - *Compute* $J := \text{in}_\omega(I)$
  - *If J contains no monomial, then output C*

Gröbner fan VS tropical variety

Let I be the ideal generated by the 3x3 minors of a 4x4 matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}$$

in the polynomial ring of 16 variables.

- The Gröbner fan has 163032 full-dimensional cones.
- T(I) is a 12-dimensional subfan with 936 maximal cones.

We do not want to compute the entire Gröbner fan.

Gfan

Gfan (Jensen, 2005-present):
software for computing Gröbner fans and tropical varieties.

Among others Gfan can compute the following:

1. Gröbner fans
2. Intersections of tropical hypersurfaces
3. Tropical varieties of prime ideals.

- Algorithms appeared in
  [Fukuda, Jensen, Thomas]
  [Bogart, Jensen, Thomas, Speyer, Sturmfels]

Tropical varieties of prime ideals

A polyhedral fan is *pure* of dimension d if all maximal cones have dimension d.

Theorem (Bieri-Groves, 1984)

Let I be a monomial-free prime ideal.The tropical variety $T(I)$ is pure of dimension Krull $\dim(\mathbb{C}[x_1, \ldots, x_n]/I)$.

Notice

- $T(I \cap J) = T(I) \cup T(J)$
- $T(I) = T(\sqrt{I})$.

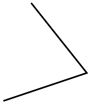
Primary decomposition gives:

Corollary

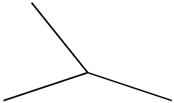
Every tropical variety is the finite union of pure tropical varieties.

Balancing property

Every pure dimensional tropical variety $T(I)$ is balanced.



Unbalanced



Balanced

Can Gröbner bases be avoided?

- Given $I = \langle f_1, \ldots, f_m \rangle$, assume that coefficients are generic. Deciding if $\omega \in T(I)$ can be done by a Mixed Volume computation.

  More generally:
  Allermann and Rau define tropical varieties as balanced fans and work completely with polyhedral constructions (tropical intersection theory).

What we will do:

- Do the tropical hypersurface intersection, and spot the tropical varity inside.

Determining the dimension of a variety $V(I)$

Suppose we cannot compute a Gröbner basis of $I = \langle f_1, \ldots, f_r \rangle$.

- Then we cannot compute $T(I)$,
- but we can compute the superset

$$\bigcap_i T(\langle f_i \rangle) \supseteq T(I).$$

The properties

- $T(I)$ is balanced.
- A projection of a $T(I)$ is a tropical variety.
- A tropical variety in $\mathbb{R}^1$ is either $\emptyset$, $\{0\}$, or $\mathbb{R}^1$.
- $T(I)$ can be decomposed into pure tropical varieties.

can be used to bound the dimension of $T(I)$ (and $V(I)$).

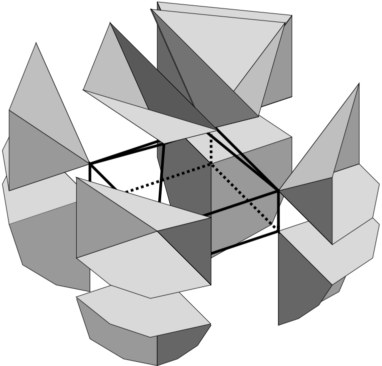
Tropical hypersurfaces

Algorithm

Input *A polynomial* $f \in \mathbb{C}[x_1, \ldots, x_n]$

Output *A collection of cones* $T(\langle f \rangle)$

- *Compute the normal fan of the Newton polytope of* f.
- *Take only those cones of dimension* $n - 1$.



Intersections of tropical hypersurfaces

Algorithm

Input *Polynomials* $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$

Output *A fan representing* $\bigcap_i T(\langle f_i \rangle)$

- *Compute* $T(\langle f_1 \rangle), \ldots, T(\langle f_r \rangle)$
- *Repeatedly apply*

$$A \wedge B := \{a \cap b : a \in A, b \in B\}$$

to get $T(\langle f_1 \rangle) \wedge \cdots \wedge T(\langle f_r \rangle)$.

An example in celestial mechanics

Joint work in progress with Marshall Hampton:
We have 47 equations $f_1, \dots, f_{47} \in \mathbb{C}[x_1, \dots, x_{10}]$ generating I.

We wish to determine $\dim(V(I))$ inside $(\mathbb{C}^*)^n$, but we cannot compute a Gröbner basis.

We may easily compute

$$\bigcap_i T(\langle f_i \rangle) \supseteq T(I)$$

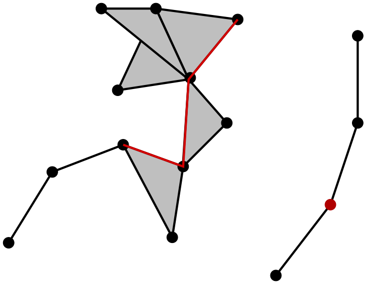
This is a fan with 117 cones up to symmetry.

We wish to show that the right hand side is zero-dimensional.
For each of the $117 - 1$ cones we wish to compute $\mathrm{in}_\omega(I)$ and show that it contains a monomial.

We can only compute $J_\omega := \langle \mathrm{in}_\omega(f_1), \dots, \mathrm{in}_\omega(f_{47}) \rangle$.

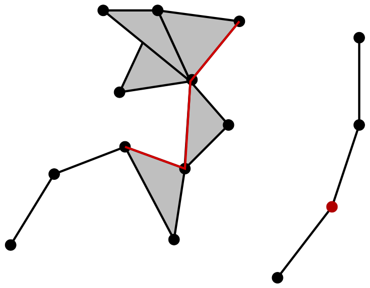
How can we use properties of tropical varieties to argue about dimensions?

$T(I)$ is contained in



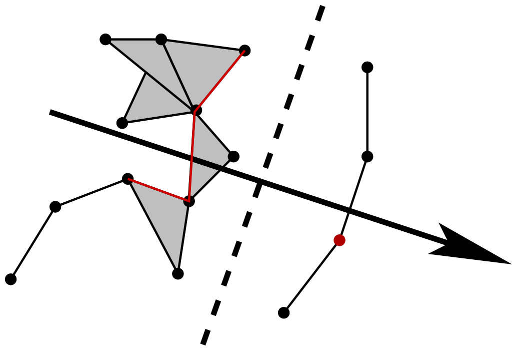
Drawing is projective and up to an S_5-symmetry.
$\dim(T(I)) \leq 3$.

Balancing property



- The 2-dimensional red cones are not balanced.
- $\Rightarrow$ the three adjacent 3-dimensional cones cannot contain 3-dimensional stuff from $T(I)$.
- $\Rightarrow$ The “center” 2-dimensional cone cannot be balanced.
- $\Rightarrow$ $\dim(T(I)) \leq 2$.

Projection property



- Not balanced at the red ray.
- $\Rightarrow$ right hand side is at most one-dimensional.

Decomposition, projection $\Rightarrow \dim(T(I)) \leq 1$

References

- Bieri, Groves: “The geometry of the set of characters induced by valuations” (1984)
- Mora, Robbiano: “The Gröbner fan of an ideal” (1988)
- Collart, Kalkbrener, Mall: ”Converting bases with the Gröbner walk” (1993)
- Speyer, Sturmfels: “The tropical Grassmannian” (2004)
- Hampton, Moeckel:“Finiteness of relative equilibria of the four-body problem” (2006)

- Bogart, J., Thomas, Speyer, Sturmfels: “Computing tropical varieties” (2007)
- Hampton, Jensen: “Finiteness of spatial central configurations...” (in preparation)