Incomplete $\mathcal{A}$-Hypergeometric Systems

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Overview

"Incomplete $\mathcal{A}$-Hypergeometric Systems"

(arXiv:0907.0745 [math.CA])

- Generalization of the theory of $\mathcal{A}$-hypergeometric system
    - Definition of incomplete $\mathcal{A}$-hypergeometric system
    - For incomplete $\boldsymbol{\Delta_1 \times \Delta_1}$-hypergeometric system,
        - Contiguity relation
        - Series solution
        - Monodromy formula
        - Connection formula

Motivation I

- JST CREST project
  keywords : statistics, Gröbner basis

- Incomplete beta function

$$B(\alpha, \beta; y) = \int_0^y s^{\alpha-1}(1-s)^{\beta-1} ds$$

- Beta function

$$B(\alpha, \beta) = \int_0^1 s^{\alpha-1}(1-s)^{\beta-1} ds$$

  - $[0, 1]$ is a cycle, but $[0, y]$ is not a cycle.

Motivation II

- Marginal Likelihood Integrals for Mixtures of Independence Models.

`arXiv:0805.3602v1 [stat.CO]`, Shaowei Lin, Bernd Sturmfels, Zhiqiang Xu

$$\int_D \prod_{i,j\in\{\mathtt{A},\mathtt{C},\mathtt{G},\mathtt{T}\}} \left(\pi\lambda_i^{(1)}\lambda_j^{(2)} + \tau\rho_i^{(1)}\rho_j^{(2)}\right)^{U_{ij}} \omega$$

$$D : \pi + \tau = 1, \pi, \tau \geq 0, \lambda_{\mathtt{A}}^{(1)} + \lambda_{\mathtt{C}}^{(1)} + \lambda_{\mathtt{G}}^{(1)} + \lambda_{\mathtt{T}}^{(1)} = 1, \lambda_{\mathtt{i}}^{(1)} \geq 0, \ldots$$

- M.A.Chaudhry, Asghar Qadir,

Incomplete Exponential and Hypergeometric Functions with Applications to the Non Central χ^2-Distribution. Communications in Statistics – Theory and Models **34**

Incomplete $\mathcal{A}$-Hypergeometric Systems I

Definition (Incomplete $\mathcal{A}$-Hypergeometric Systems)

We call the following system of differential equations $H_A(\beta, g)$ an incomplete $\mathcal{A}$-hypergeometric system: for $A = (a_{ij})$

$$
\begin{aligned}
\left(\sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i \right) \bullet f &= g_i, \qquad (i = 1, \ldots, d) \\
\left(\prod_{i=1}^{n} \partial_i^{u_i} - \prod_{j=1}^{n} \partial_j^{v_j} \right) \bullet f &= 0
\end{aligned}
$$

with $u, v \in \mathbf{N}_0^n$ running over all u, v such that $Au = Av$.

Here, $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$, and $\beta = (\beta_1, \ldots, \beta_d) \in \mathbf{C}^d$ are parameters and $g = (g_1, \ldots, g_d)$ where g_i are given holonomic functions which may depend on parameters β.

Incomplete $\mathcal{A}$-Hypergeometric Systems II

Definition

A multivalued holomorphic function g on a Zariski openset in $\mathbf{C}^n$ is holonomic $\Longleftrightarrow$ there exists $I \subset D$ such that

- D/I *:holonomic*
- $I \bullet g = 0$

Theorem

Solutions of the incomplete $\mathcal{A}$-hypergeometric system are holonomic functions.

- f : holonomic $\Longrightarrow$ there exists a homogenious system of differential equations.
- We can apply some algorithms for holonomic systems.
  - To find candidates of series solutions
  - Contiguity relation, etc....

Example I. The incomplete beta function

Example

$$B(\alpha,\beta;y) = \int_0^y s^{\alpha-1}(1-s)^{\beta-1}ds$$

Replacing s by yt, we have

$$B(\alpha,\beta;y) = y^{\alpha}\int_0^1 t^{\alpha-1}(1-yt)^{\beta-1}dt.$$

Put $B(\alpha,\beta;y) = y^{\alpha}\tilde{B}(\beta-1,\alpha-1;1,-y)$
where $\tilde{B}(\beta-1,\alpha-1;x_1,x_2) = \int_0^1 t^{\alpha-1}(x_1+x_2t)^{\beta-1}dt$.
The function $\tilde{B}$ is a solution of an incomplete $\mathcal{A}$-hypergeometric system $H_A(\beta,g)$ for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, parameter : $(\beta-1,\alpha-1)$,
$g_1 = 0, g_2 = (x_1+x_2)^{\beta-1}$.

Incomplete $\Delta_1 \times \Delta_1$-hypergeometric functions

Definition (Incomplete $\Delta_1 \times \Delta_1$-hypergeometric function)

We assume $0 < a < b$ for simplicity.

$$\Phi(\alpha_1, \alpha_2, -\gamma - 1; x) := \int_a^b t^\gamma (x_{11} + x_{21} t)^{\alpha_1} (x_{12} + x_{22} t)^{\alpha_2} dt$$

for $x_{ij} > 0$ and $\mathrm{Re}\,(\gamma), \mathrm{Re}\,(\alpha_i) > -1$.

The integral and its analytic continuations satisfy the following incomplete $\mathcal{A}$-hypergeometric system.

$$\begin{cases} \left(\theta_{11}\theta_{22} - \frac{x_{11}x_{22}}{x_{21}x_{12}}\theta_{21}\theta_{12}\right) \bullet f & = 0 \\ (\theta_{11} + \theta_{21} - \alpha_1) \bullet f & = 0 \\ (\theta_{12} + \theta_{22} - \alpha_2) \bullet f & = 0 \\ (\theta_{21} + \theta_{22} + \gamma + 1) \bullet f & = [g(t, x)]_{t=a}^{t=b} \end{cases}$$

where, $\theta_{ij} = x_{ij}\partial_{ij}$, $g(t, x) = t^{\gamma+1}(x_{11} + x_{21} t)^{\alpha_1}(x_{12} + x_{22} t)^{\alpha_2}$.

Example II. The incomplete elliptic integral of the first kind

Example

$$F(z;k) = \int_0^z \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

Replacing x^2 by $z^2 t$, we obtain

$$F(z;k) = \frac{1}{2} z \int_0^1 t^{-\frac{1}{2}}(1-z^2 t)^{-\frac{1}{2}}(1-k^2 z^2 t)^{-\frac{1}{2}} dt.$$

The incomplete elliptic integral of the first kind can be regarded as a solution of an incomplete $\Delta_1 \times \Delta_1$-hypergeometric system

$H_A(\beta, g)$ for $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$, $\beta = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$,

$x_{11} = 1$, $x_{21} = -z^2$,$x_{21} = 1$,$x_{22} = -k^2 z^2$,

$g_1 = g_2 = 0$, $g_3 = z(1-z^2)^{-\frac{1}{2}}(1-k^2 z^2)^{-\frac{1}{2}}$.

Homogeneous system

Solutions of $\Delta_1 \times \Delta_1$-hypergeometric system satisfy the following relatively simple holonomic system.

$$\begin{cases} \left(\theta_{11}\theta_{22} - \frac{x_{11}x_{22}}{x_{21}x_{12}}\theta_{21}\theta_{12}\right) \bullet f & = 0 \\ (\theta_{11} + \theta_{21} - \alpha_1) \bullet f & = 0 \\ (\theta_{12} + \theta_{22} - \alpha_2) \bullet f & = 0 \\ (\partial_{22} - a\partial_{21})(\partial_{12} - b\partial_{11})(\theta_{21} + \theta_{22} + \gamma + 1) \bullet f & = 0 \end{cases}$$

$$(\theta_{21} + \theta_{22} + \gamma + 1) \bullet f = [g(t,x)]_{t=a}^{t=b}$$

$$(\partial_{22} - a\partial_{21})(\partial_{12} - b\partial_{11}) \bullet [g(t,x)]_{t=a}^{t=b} = 0$$

Series solutions I

We define the following 4 series.

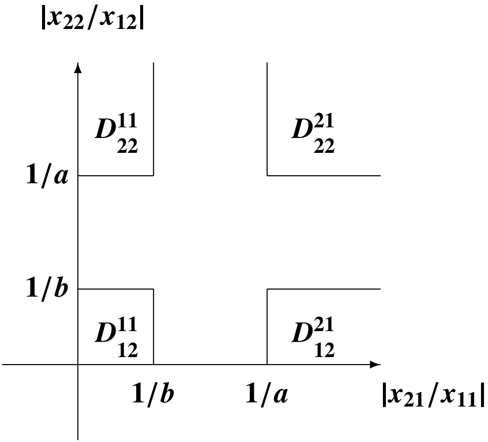
$$f_{12}^{11} = x_{11}^{\alpha_1} x_{12}^{\alpha_2} \sum_{k,m \geq 0} \frac{(-1)^{k+m}}{\gamma + k + m + 1} \cdot \frac{(-\alpha_1)_k (-\alpha_2)_m}{(1)_k (1)_m} \cdot (b^{\gamma+k+m+1} - a^{\gamma+k+m+1}) \left(\frac{x_{21}}{x_{11}}\right)^k \left(\frac{x_{22}}{x_{12}}\right)^m$$

$$f_{22}^{11} = x_{11}^{\alpha_1} x_{22}^{\alpha_2} \sum_{k,m \geq 0} \frac{(-1)^{k+m}}{\gamma + \alpha_2 + k - m + 1} \cdot \frac{(-\alpha_1)_k (-\alpha_2)_m}{(1)_k (1)_m} \cdot (b^{\gamma+\alpha_2+k-m+1} - a^{\gamma+\alpha_2+k-m+1}) \left(\frac{x_{21}}{x_{11}}\right)^k \left(\frac{x_{12}}{x_{22}}\right)^m$$

$$f_{12}^{21} = x_{21}^{\alpha_1} x_{12}^{\alpha_2} \sum_{k,m \geq 0} \frac{(-1)^{k+m}}{\gamma + \alpha_1 - k + m + 1} \cdot \frac{(-\alpha_1)_k (-\alpha_2)_m}{(1)_k (1)_m} \cdot (b^{\gamma+\alpha_1-k+m+1} - a^{\gamma+\alpha_1-k+m+1}) \left(\frac{x_{11}}{x_{21}}\right)^k \left(\frac{x_{22}}{x_{12}}\right)^m$$

$$f_{22}^{21} = x_{21}^{\alpha_1} x_{22}^{\alpha_2} \sum_{k,m \geq 0} \frac{(-1)^{k+m}}{\gamma + \alpha_1 + \alpha_2 - k - m + 1} \cdot \frac{(-\alpha_1)_k (-\alpha_2)_m}{(1)_k (1)_m} \cdot (b^{\gamma+\alpha_1+\alpha_2-k-m+1} - a^{\gamma+\alpha_1+\alpha_2-k-m+1}) \left(\frac{x_{11}}{x_{21}}\right)^k \left(\frac{x_{12}}{x_{22}}\right)^m$$

Convergence domains of $f_{ij}^{k\ell}$



Series solutions II

Theorem

We assume that $\gamma \notin \mathbf{Z}$.

1. *The series* $f_{ij}^{k\ell}$ *converges on the domain* $D_{ij}^{k\ell}$.
2. *The function* $f_{ij}^{k\ell}$ *satisfies the incomplete* $\Delta_1 \times \Delta_1$*-hypergeometric system for a branch of* $[g(t,x)]_{t=a}^{t=b}$.
3. f_{12}^{11} *can be expressed in terms of the Appell function* F_1 *as*

$$
\begin{aligned}
f_{12}^{11} \quad = \quad & x_{11}^{\alpha_1} x_{12}^{\alpha_2} \left(\frac{b^{\gamma+1}}{\gamma+1} F_1\left(\gamma+1, -\alpha_1, -\alpha_2, \gamma+2, \frac{-x_{21}b}{x_{11}}, \frac{-x_{22}b}{x_{12}}\right) \right. \\
& \left. - \frac{a^{\gamma+1}}{\gamma+1} F_1\left(\gamma+1, -\alpha_1, -\alpha_2, \gamma+2, \frac{-x_{21}a}{x_{11}}, \frac{-x_{22}a}{x_{12}}\right)\right)
\end{aligned}
$$

Example III

For the incomplete elliptic integral of the first kind:

$$F(z;k) = \frac{1}{2} z \int_0^1 t^{-\frac{1}{2}}(1-z^2t)^{-\frac{1}{2}}(1-k^2z^2t)^{-\frac{1}{2}} dt,$$

let us apply Theorem to obtain an expression of $F(z;k)$ in terms of the Appell function F_1.

Example

Put $x_{11} = 1, x_{21} = -z^2, x_{12} = 1, x_{22} = -k^2z^2$ and $\alpha_1 = \alpha_2 = \gamma = -\frac{1}{2}, a = 0, b = 1$. Then we have

$$F(z;k) = \frac{1}{2} z \cdot \frac{1}{-\frac{1}{2}+1} F_1\left(-\frac{1}{2}+1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}+2; \frac{z^2}{1}, \frac{k^2z^2}{1}\right)$$

$$= zF_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2, k^2z^2\right).$$

(http://functions.wolfram.com/08.05.26.0006.01)

Future studies

- Incomplete $\Delta_1 \times \Delta_n$-hypergeometric system
- Incomplete $\Delta_2 \times \Delta_2$-hypergeometric system

Future studies

- Incomplete $\Delta_1 \times \Delta_n$-hypergeometric system
- Incomplete $\Delta_2 \times \Delta_2$-hypergeometric system

Thank You !!