CONES OF HILBERT FUNCTIONS

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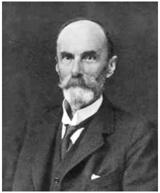
Fundamental Problems

Fix $S := \Bbbk[x_0, \ldots, x_n]$ where $\deg x_i = 1$. Let M be an $\mathbb{N}$-graded S-module.

HILBERT (1890): If $h_M : \mathbb{N} \to \mathbb{N}$ is defined by $h_M(i) := \dim_{\Bbbk} M_i$, then there exists $p_M \in \mathbb{Q}[i]$ such that $h_M(i) = p_M(i)$ for $i \gg 0$.

PROBLEMS: (1) For a collection of modules, describe the set (space?) of all Hilbert functions.
(2) Describe the space of all modules with a given Hilbert function.

Prototype Solution to (1)



For positive integers m and i, there is an expression $m = \binom{k_i}{i} + \binom{k_{i-1}}{i-1} + \cdots + \binom{k_j}{j}$ with $k_i > k_{i-1} > \cdots > k_j \geq j \geq 1$.

Define $m^{\langle i \rangle} := \binom{k_i+1}{i+1} + \binom{k_{i-1}+1}{i} + \cdots + \binom{k_j+1}{j+1}$.

Macaulay (1927): For $h : \mathbb{N} \to \mathbb{N}$, the following are equivalent:

(a) $h(0) = 1$ and $h(i+1) \leq h(i)^{\langle i \rangle}$ for $i \geq 1$;

(b) there exists a homogeneous ideal I such that $h(i) = h_{S/I}(i)$ for all $i \in \mathbb{N}$.

Impact of Macaulay's Work

STRENGTHS: **Proof distinguishes the *lex-segment ideal*. This monomial ideal has extremal syzygies and plays a central role in establishing the connectedness of the Hilbert scheme.**

WEAKNESSES: **(I) The function $m \longmapsto m^{(i)}$ is cumbersome.**
(II) Analogues of the lex-segment ideal fail to exist in many similar situations.

Cone of Hilbert Functions

- The set of Hilbert functions forms a semigroup: $h_{M\oplus N}(i) = h_M(i) + h_N(i)$.
- If we bound $a := \max\{i : h_M(i) \neq p_M(i)\}$, then this is an affine semigroup (i.e. subsemigroup of $\mathbb{Z}^{n+a+1}$).
- PROBLEM: Describe the convex hull of the set of Hilbert functions.
- REMARK: In general, this semigroup is not saturated — there exists lattice points in the convex hull which do not correspond to Hilbert functions.

Preferred Basis

HILBERT REFORMULATED:

$F_M(t) := \sum_{j\geq 0} h_M(i)t^i \in \mathbb{Q}(t)$ and $\deg F_M(t) = \max\{i : h_M(i) \neq p_M(i)\}$.

Set $\Delta h(i) := h(i+1) - h(i)$ for all $i \in \mathbb{Z}$.

LEMMA: If M is a finitely generated $\mathbb{N}$-graded S-module, $\dim M = d$, and $a \geq \deg F_M(t)$ then

$$F_M(t) = \sum_{i=0}^{a} h_M(i)t^i + t^a \sum_{j=0}^{d-1} \Delta^j h(a+1) \frac{t^{j+1}}{(1-t)^{j+1}}.$$

COROLLARY: We have

$$h \longleftrightarrow \big(h(0), \ldots, \Delta^{d-1} h(a+1)\big) \in \mathbb{Z}^{d+a+1}.$$

Facet Inequalities

Theorem (Boij-Smith): If M is an $\mathbb{N}$-graded S-module that is finitely generated in degree 0, $\dim M = d$, and $a \geq \deg F_M(t)$, then $h_M \in \mathbb{Z}^{d+a+1}$ lies in the rational simplicial cone defined by the half-spaces:

$$\frac{h_M(i)}{\binom{n+i}{i}} \geq \frac{h_M(i+1)}{\binom{n+i+1}{i+1}} \qquad \text{for } 0 \leq i \leq a$$

$$\frac{\Delta^j h_M(a+1)}{\binom{n+a+1}{j+a+1}} \geq \frac{\Delta^{j+1} h_M(a+1)}{\binom{n+a+1}{j+a+2}} \qquad \text{for } 0 \leq j < d.$$

Extremal Rays

Let $R(d,a) := S/\langle x_d,\ldots,x_n\rangle^{d+a+1}$ so

$$F_{R(d,a)}(t) = \sum_{i=0}^{a}\binom{n+i}{i}t^i + t^a\sum_{j=0}^{d-1}\binom{n+a+1}{j+a+1}\frac{t^{j+1}}{(1-t)^{j+1}}.$$

Theorem (Boij-Smith): If M is an $\mathbb{N}$-graded S-module that is finitely generated in degree 0, $\dim M = d$, and $a \geq \deg F_M(t)$, then $h_M \in \operatorname{pos}_{\mathbb{Q}}(h_{R(d,a)},\ldots,h_{R(0,a)},\ldots,h_{R(0,0)})$.

Multigraded Version

Fix a smooth projective toric variety X with Cox ring S; $\deg x_\rho := \mathcal{O}(D_\rho) \in \operatorname{Pic} X \cong \mathbb{Z}^r$. Smoothness means $h : \operatorname{Psef} X \to \mathbb{N}$ is a polynomial on a translate of $\operatorname{Nef} X$.

Let $B = \cap P_\sigma$ where $P = \langle x_\rho : \rho \in \sigma \rangle$ be its irrelevant ideal; subschemes $Y \subseteq X$ correspond to B-saturated S-ideals.

Skeleton of a Theorem: The cone of Hilbert functions for $Y \subseteq X$ is generated by $S / \cap P_\tau^e$ where $\tau \subseteq \sigma$.