# Tutorial for CMC-Lab

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## 1 Introduction

In this note, we will give a instruction for CMC-Lab software. CMC-Lab was programmed by Nicolas Schmitt for a research of constant mean curvature surfaces around 2000 – 2001.

## 2 Installation of CMC-Lab

One can download CMC-Lab software from the following web page:

1. **Linux version** (Redhat, Debian etc...)
   ⇒ http://www.gang.umass.edu/software/cmclab/index.html

2. **Java version**
   ⇒ http://tmugs.math.metro-u.ac.jp/javacmclab030926.zip

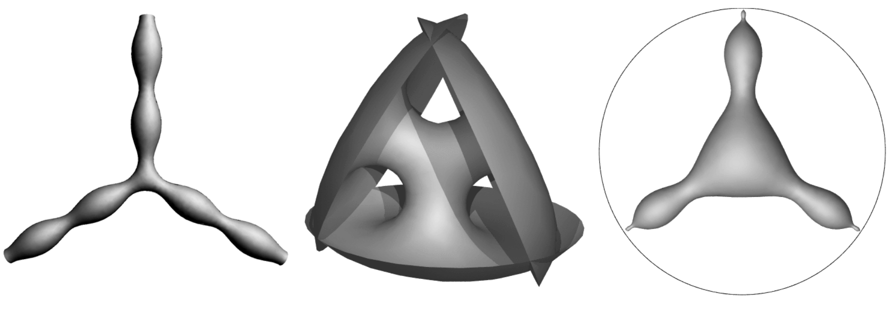

Figure 1: CMC bubbletons in $\mathbb{R}^3$, $S^3$ and $H^3$.

The detailed instruction for the installation of CMC-Lab is given in

- Installation guide for linux version
  http://www.math.sci.kobe-u.ac.jp/~ kobayasi/GPS/tex_html_files/GPSCMCLab/
- Linux version is more advantageous than java version, however java version is the only choice for windows users.

# 3 Dorfmeister-Pedit-Wu method

In this section, we will give a brief explanation of theory of Dorfmeister, Pedit, Wu ([2]) to construct CMC surfaces, which is used for the CMC-Lab software.

First, we identify $\mathbb{R}^3$ and $su(2) = \mathrm{Im}\mathbb{H}$ as follows, where $\mathbb{H}$ is the quaternion.

$$\mathbb{R}^3 \Longleftrightarrow su(2) = \left\{ A \in \mathrm{Mat}(2, \mathbb{C}) \; ; \; \bar{A}^t = -A \right\} \quad .$$

Therefore, for example, we have the correspondence between $\mathbb{R}^3$ and $su(2)$ as follows:

$$\text{Adjoint group actions on } su(2) \text{ by } SU(2) \overset{2:1}{\Longleftrightarrow} \text{Rotations of } \mathbb{R}^3 \text{ by } SO(3).$$

Now we give a brief explanation of Dorfmeister, Pedit, Wu methods. The methods can be divided as the following 4 steps. Details can be found in [2] and [3].

**Step1** : Let $\mathcal{D}$ be a simply connected domain in $\mathbb{C}$.

- ○ $\eta(z, \lambda) = \sum_{n=-1}^{\infty} A_n \lambda^n dz.$
- ○ $2 \times 2$ matrix differential form and Tr $\eta = 0$.
- ○ diagonal even in $\lambda$, off-diagonal odd in $\lambda$.
- ○ $A_j$ are holomorphic with respect to $z \in \mathfrak{D}$.
- ○ $\det A_{-1} \neq 0$.

**Step2** : Solve the ODE $dC = C\eta$.

**Step3** : Iwasawa decomposition: $C = FW_+$

- $\circ$ $F = F(z, \bar{z}, \lambda)$ is unitary for all z $\in \mathfrak{D}, \lambda \in \mathbb{S}^1$.
- $\circ$ $W_+ = \displaystyle\sum_{n=0}^{\infty} W_{n,+}\lambda^n$.

**Step4** : (Sym-Bobenko-Formula)

$$\Psi_\lambda(z) = -\frac{1}{2H}\left\{\left(i\lambda\frac{d}{d\lambda}F\right)F^{-1} + F\frac{i}{2}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)F^{-1}\right\}$$

$$\Rightarrow \begin{cases} \Psi_\lambda \text{ is a CMC-immersion from } \mathfrak{D} \text{ to } \mathbb{R}^3. \\ \text{Every CMC-immersion can be obtained this way.} \end{cases}$$

In fact, the solution $C$ is in loop group of $SL(2, \mathbb{C})$, which is a infinite dimensional Lie group. We do not give the definitions of loop groups here, and refer the article [2] to readers. Analogously $F$ is in loop group of $SU(2)$, and $W_+$ is in plus loop group of $SL(2, \mathbb{C})$. We also refer the article [1] to readers for “Sym-Bobenko formula” in Step 4. We use the notation $\Lambda SL(2, \mathbb{C})$ (resp. $\Lambda SU(2)$ and $\Lambda^+ SL(2, \mathbb{C})$) for the loop group of $SL(2, \mathbb{C})$ (resp. loop group of $SU(2)$ and the plus loop group of $SL(2, \mathbb{C})$).

# 4 Algorithm for CMC-Lab

For the implementation of Dorfmeister, Pedit, Wu method, there are two main issues, which are Step 2 and Step3 in the previous section. In Step2, $dC = C\eta$ is a first order $2\times2$ matrix differential equation. Thus we have many algorithms, for example Runge-Kutta method. Therefore we concentrate the algorithm for Step 3, which is Iwasawa decomposition. We quote the following lemma from [4].

**Lemma 1** *Set*

$$W = \text{span}\{C^1, \lambda C^1, \cdots, C^2, \lambda C^2, \cdots\}\ .$$

*Let* $C \in \Lambda SL_2(\mathbb{C})$*, and* $C^1$*,* $C^2$ *be the columns of* $C$*. If* $x, y \in W \cap (\lambda W)^\perp$*, then*

$$\langle x, y\rangle_{\mathbb{C}^2} = \langle x, y\rangle_H \ \ \textit{and}\ \ \dim(W \cap (\lambda W)^\perp) = 2\ \ ,$$

*where*

$$\langle x, y\rangle_H = \frac{1}{2\pi i}\int_{C_r} \langle x, y\rangle_{\mathbb{C}^2}\frac{d\lambda}{\lambda}\ .$$

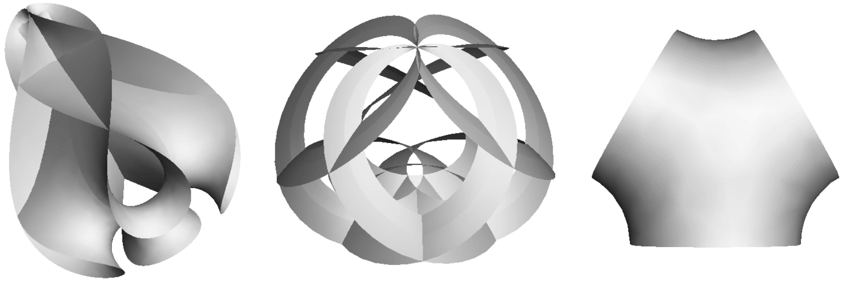

Figure 2: genus one CMC surfaces (the left two pictures) and a periodic CMC surface (the right picture).

Then we will state the main theorem.

**theorem 2** *Set*

$$P^j : C^j \to \lambda W \ \ \textit{(projection to } \lambda W\textit{)}\ .$$

*and*

$$P = (P^1, P^2)\ .$$

*Then $P = CB_+$ for some loop $B_+$ with positive Fourier terms. Set*

$$G = (G^1, G^2) = C - P\ \ .$$

*Take unitary part of $G$ via Hilbert norm, that is, $G = FB_0$*

$$B_0 = \begin{pmatrix} |G^1| & \langle G^2, G^1/|G^1|\rangle \\ 0 & |G^2 - G^1/|G^1|\langle G^2, G^1/|G^1|\rangle| \end{pmatrix}\ \ .$$

*Then $C = F \cdot B_0(I - B_+)^{-1}$ is the Iwasawa decomposition of $C$.*

**Proof 1** *Clearly, $G$ is in $W \cap (\lambda W)^\perp$, thus Lemma 1 implies that the columns $G^1$ and $G^2$ are the basis of $W \cap (\lambda W)^\perp$. Then we can do the Gram-Schmidt orthogonalization for $G$ in $\mathbb{C}^2$.* □

Theorem 2 implies that if one can find the projection $P$, then one can compute the Iwasawa decomposition.

## 4.1 Algorithm of Step 3

In this subsection, we will give the algorithm for Step 3 in previous section. Next lemma is important for a computation of the projection $P$ defined in previous section.

**Proposition 3** *Set*

$$\mathcal{A} = \{a_1, \cdots, a_n\} \quad : \textit{a basis for } \mathbb{C}^n.$$

*Take* $0 \leq r \leq n$,

$$p : \mathbb{C}^n \to \mathbb{C}^n \quad : \textit{projection to the subspace spanned by } \{a_1, \cdots, a_r\},$$

$$A = (a_1, \cdots, a_n) \in \mathrm{SL}(2, \mathbb{C}) \quad ,$$

*and*

$$\tilde{P} = \begin{pmatrix} I_r & 0 \\ 0 & O_{n-r} \end{pmatrix} \in Mat(n, \mathbb{C}) \ .$$

*Then* $p$ *can be written as follows:*

*1* $A\tilde{P}A^{-1}$,

*2* $U\tilde{P}\bar{U}^t$, *where* $A = UT$ *is the QR-decomposition of* $A$.

**Proof 2** *The matrix* $\tilde{P}$ *is the projection to the subspace spanned by* $\{e_1, \cdots, e_r\}$ *of the space spanned by the standard basis* $\{e_1, \cdots, e_n\}$ *for* $\mathbb{C}^r$. *Therefore one can write the projection* $p$ *to the subspace spanned by* $\{a_1, \cdots, a_r\}$ *of the space spanned by* $\{a_1, \cdots, a_n\}$ *for* $\mathbb{C}^n$ *as* 1. *The matrix* $T$ *commute* $\tilde{P}$, *thus* $A\tilde{P}A^{-1} = UT\tilde{P}T^{-1}U^{-1} = U\tilde{P}U^{-1}$. *And* $U$ *is unitary implies that* $U^{-1} = \bar{U}^t$. □

Computing a inverse matrix takes long time for a numerical computation. Therefore we will use the expression 2 of Proposition 3 as the projection $p$.

Now we will apply Proposition 3 for the actual object. We take a finite part of $\tilde{A} \in \Lambda SL(2,\mathbb{C})$ as follows:

$$A = \begin{pmatrix} \Sigma_{k=-n}^{n} a_k^{11}\lambda^k & \Sigma_{k=-n}^{n} a_k^{12}\lambda^k \\ \Sigma_{k=-n}^{n} a_k^{21}\lambda^k & \Sigma_{k=-n}^{n} a_k^{22}\lambda^k \end{pmatrix} \in SL(2,\mathbb{C}).$$

Set $r$ is even, $r/2 \leq n$,

$$a_1 = \begin{pmatrix} \Sigma_{k=-n}^{n} a_k^{11}\lambda^k \\ \Sigma_{k=-n}^{n} a_k^{21}\lambda^k \end{pmatrix} \quad , \quad a_2 = \begin{pmatrix} \Sigma_{k=-n}^{n} a_k^{12}\lambda^k \\ \Sigma_{k=-n}^{n} a_k^{22}\lambda^k \end{pmatrix}$$

and

$$\lambda W = \operatorname{span}\left\{\lambda a_1, \cdots, \lambda^{r/2} a_1, \lambda a_2, \cdots, \lambda^{r/2} a_2\right\} \ .$$

Then the projection $p$ can be computed by Proposition 3 as follows:

$$(U_0, 0)\tilde{P}\overline{(U_0, 0)}^t,$$

where $(A_0, 0) = (U_0, 0)\begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix}$ is QR-decomposition of $A_0$.

$$A_0 = \left(\begin{array}{cccc|cccc}
0 & & & & 0 & & & \\
a_{-n}^{11} & & & & a_{-n}^{12} & & & \\
\vdots & a_{-n}^{11} & & & \vdots & a_{-n}^{12} & & \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \\
\vdots & \vdots & & a_{-n}^{11} & \vdots & \vdots & & a_{-n}^{12} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_{n-1}^{11} & a_{n-2}^{11} & \cdots & a_{n-r/2}^{11} & a_{n-1}^{12} & a_{n-2}^{12} & \cdots & a_{n-r/2}^{12} \\
\hline
0 & & & & 0 & & & \\
a_{-n}^{21} & & & & a_{-n}^{22} & & & \\
\vdots & a_{-n}^{21} & & & \vdots & a_{-n}^{22} & & \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \\
\vdots & \vdots & & a_{-n}^{21} & \vdots & \vdots & & a_{-n}^{22} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_{n-1}^{21} & a_{n-2}^{21} & \cdots & a_{n-r/2}^{21} & a_{n-1}^{22} & a_{n-2}^{22} & \cdots & a_{n-r/2}^{22}
\end{array}\right)$$

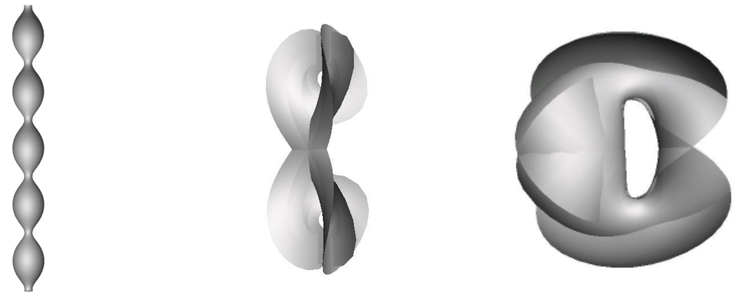

Figure 3: A CMC surface of revolution (the left picture) and CMC cylinders (the right pictures).

# 5 Some remarks

- **1984,** D. Hoffman started to use computer graphics for studying surfaces. (W. Meeks and he proved the embeddedness of Costa minimal surface [5].)

- **1998,** D. Lerner and I. Sterling made the first implementation of Dorfmeister-Pedit-Wu method [6].

# 6 Related softwares

- JavaView (which is used for a visualization of java version CMC-Lab). http://www.javaview.de/

- GeomView (which is a graphics viewer corresponding to various formats). http://www.geomview.org/

- Mesh (which construct minimal surfaces). http://www.msri.org/publications/sgp/jim/software/

- Surface evolver (which is a visualization tool for surfaces using variational problems). http://www.susqu.edu/facstaff/b/brakke/evolver/

# References


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[3] J. Dorfmeister, G. Haak, *Construction of non-simply connected CMC surfaces via dressing*, J. Math. Soc. Japan **55** (2003), no. 2, 335–364.

[4] S. Fujimori, S-P. Kobayashi, W. Rossman, *A basic introduction to loop group methods for constant mean curvature surfaces*, http://www.math.sci.kobe-u.ac.jp/HOME/wayne/wayne-e.html.

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[7] U. Pinkall and I. Sterling, *On the classification of constant mean curvature tori*, Annals of Math. **130**, 407-451 (1989).

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