(Some) computable objects in D-modules theory

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A D–module is a module over the ring D.

It represents a system of LPDE.

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Theory developed (from 1970) by I.N. Bernstein, M. Kashiwara, T. Kawai, B. Malgrange, Z. Mebkhout, D. Quillen, M. Sato and others.

Linear Partial Differential Equations

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The system of LPDE

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has no non-zero holomorphic solution (at the origin).

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What does it look like the set of LPDO $Q=Q(x,y,\partial_x,\partial_y)$ such that $Q(\frac{1}{xy})=0$?

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A kind of “inverse problem": The input is the solution $\frac{1}{xy}$ and we want the set of equations $Q(x,y,\partial_x,\partial_y)(u(x,y))=0$ having $u(x,y)=\frac{1}{xy}$ as a solution.

Problem setting: algebra tools

$x = (x_1, \ldots, x_n)$ indeterminates ($n \in \mathbb{Z}_{\geq 1}$)
$\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ polynomial ring.

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$$\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} = \frac{\partial^{\beta_1 + \cdots + \beta_n}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$$

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A_n is a (non-commutative) ring (the Weyl algebra).

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Input: A non zero polynomial $f \in \mathbb{C}[x]$.

Output: A finite generating system for the ideal $Ann(\frac{1}{f})$.

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Object $Ann(\frac{1}{f})$ is computable.

Oaku-Takayama's algorithm is implemented in `Kan/sm1 (risa/asir)`; `Macaulay2 (D-modules.m2)`; `Singular`.

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Ex.: $f = xyz(x+y)(x+z)(y+z)(x+y+z)$.

`Macaulay 2: RatAnn f` computes $Ann(\frac{1}{f})$. But for this example, *in my computer*, `Macaulay2` gives `*** out of memory, exiting ***`.

Nevertheless

Nevertheless, we can prove that $Ann(\frac{1}{f})$ is generated by the three operators

$$P_1, P_2, P_3$$

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$$P_1 = x\partial_x + y\partial_y + z\partial_z + 7$$

$$P_2 = y(x+y)(y+z)\partial_y - z(x+z)(y+z)\partial_z + (y-z)(x+4y+4z)$$

$$P_3 = y(x-y)(x+y)\partial_y + z(x+z)(x+3y+3z)\partial_z + 3x^2 + 5xy - 4y^2 + 8xz + 8yz + 8z^2$$

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How to prove that?

First step to $Ann(\frac{1}{f})$: order 1 operators

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If $f \in \mathbb{C}$ (and $f \neq 0$) then $Ann(\frac{1}{f}) = A_n(\partial_1, \ldots, \partial_n)$.

First step to $Ann(\frac{1}{f})$: order 1 operators

Assume f is not a constant polynomial.

First step to $Ann(\frac{1}{f})$: order 1 operators

Assume P is a first order operator

$P = \sum_{i=1}^{n} p_i(x)\partial_i + p_0(x)$

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(K. Saito): The vector field $\sum p_i(x)\partial_i$ is called *logarithmic* w.r.t. f.

Ex.: $f\partial_i$ is a logarithmic vector field (for $i = 1, \ldots, n$) w.r.t. f and $f\partial_i + \partial_i(f)$ annihilates $\frac{1}{f}$.

Logarithmic vector fields

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for some $p_0(x) \in \mathbb{C}[x]$.

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Denote $Ann^{(1)}(\frac{1}{f})$ the ideal in A_n generated by LPDO P of order 1 and $P(\frac{1}{f}) = 0$.

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$$Ann^{(1)}(\tfrac{1}{f}) \subset Ann(\tfrac{1}{f})$$

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Notice that $p_0(x) = \frac{\delta(f)}{f}$.
$\widetilde{Der}(\log f) = \{\delta + \frac{\delta(f)}{f} \mid \delta \in Der(\log f)\}$

Denote $Ann^{(1)}(\frac{1}{f})$ the ideal in A_n generated by LPDO P of order 1 and $P(\frac{1}{f}) = 0$.

Problem 2. Describe (characterize) the class of nonzero $f \in \mathbb{C}[x]$ such that
$$Ann^{(1)}(\frac{1}{f}) = Ann(\frac{1}{f}).$$

First examples

Ex.: $n = 1$, $x = x_1$.

$$Ann^{(1)}(\frac{1}{x}) = Ann(\frac{1}{x}) = A_1(x\partial_x + 1).$$

First examples

Ex.: $n = 2$, $x = x_1, y = x_2$.

$$Ann^{(1)}(\frac{1}{xy}) = Ann(\frac{1}{xy}) = A_2(x\partial_x + 1, y\partial_y + 1).$$

First examples

Ex.: $n = 2$, $x = x_1, y = x_2$.

$Ann^{(1)}(\frac{1}{x-y^2}) = Ann(\frac{1}{x-y^2}) =$
$A_2(2y\partial_x + \partial_y, (x - y^2)\partial_x)$.

First examples

Ex.: $n = 2$,

$$Ann^{(1)}\left(\frac{1}{x^4+y^5+xy^4}\right) \subsetneqq Ann\left(\frac{1}{x^4+y^5+xy^4}\right).$$

$Der(\log f)$ and syzygies

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Previous map is an isomorphism of $\mathbb{C}[x]$–modules. So, object $Der(\log f)$ is computable.

By using commutative Groebner basis computation in the polynomial ring $\mathbb{C}[x]$.

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By using commutative Groebner basis computation in the polynomial ring $\mathbb{C}[x]$.

$Ann^{(1)}(\frac{1}{f})$ is computable (using *only* commutative Groebner bases algorithms; which also have double exponential complexity).

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In practice $Ann^{(1)}(\frac{1}{f})$ is easier to compute than $Ann(\frac{1}{f})$.

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$k \in \mathbb{Z}_{\geq 1}$. $Ann^{(k)}(\frac{1}{f})$
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Ex.: $P = \sum_{i \leq j} p_{ij}(x)\partial_i\partial_j + \sum_i p_i(x)\partial_i + p_0(x)$
$P(\frac{1}{f}) = 0$ if and only if
the coefficients $(p_{ij}(x), p_i(x), p_0(x))$ represent a syzygy among f^2 and a set of expressions in the partial derivatives of f up to degree 2.

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$Ann^{(1)}(\frac{1}{f}) \subset Ann^{(2)}(\frac{1}{f}) \subset \cdots \subset Ann^{(k)}(\frac{1}{f}) \subset \cdots \subset Ann(\frac{1}{f})$.

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Problem 3. Describe the behavior of the function
$0 \neq f \in \mathbb{C}[x] \mapsto k(f)$.

Singularities Theory tools

From now on, we assume f is a reduced nonzero polynomial in $\mathbb{C}[x]$.

Ω^p differential p-forms with polynomial coefficients, $p \in \mathbb{N}$.

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$\Omega^p(1/f)$ meromorphic differential p-forms with poles along $f=0$, $p \in \mathbb{N}$.

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(E. Brieskorn) The cohomology of $\Omega^\bullet(1/f)$ is computable if f is an *arrangement of hyperplanes*.

(T. Oaku, N.Takayama) For any nonzero polynomial $f \in \mathbb{C}[x]$, the cohomology of $\Omega^\bullet(1/f)$ is computable.

Singularities Theory tools

$\Omega^p(1/f) \supset \Omega^p(\log f)$ logarithmic differential p-forms (w.r.t. f).

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Ex.: $\frac{dx}{x}$ and $\frac{dy}{y}$ are logarithmic 1-forms (w.r.t. $f = xy$).
$\frac{dx}{x^2}$, $\frac{dx}{y}$ are not.

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(N. Takayama- F.J.C.J.) Positive solution to Problem 4 if $n = 2$.

Logarithmic Comparison Theorem

Problem 5. Describe the class of nonzero polynomial f such that

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quasi-isomorphism $\equiv$ induces an isomorphism in cohomology.

Logarithmic Comparison Theorem

Problem 5. Describe the class of nonzero polynomial f such that

$$i_f : \Omega^\bullet(\log f) \to \Omega^\bullet(1/f)$$

is a quasi-isomorphism.

If so, we say that the Logarithmic Comparison Property (LCP) holds for f (or for $f = 0$).

$Ann(\frac{1}{f})$ and Log. Cohomology

(J.M. Ucha-F.J.C.J.) For (Spencer + free) polynomials $Ann^{(1)}(\frac{1}{f}) = Ann(\frac{1}{f})$ in and only if $i_f : \Omega^{\bullet}(\log f) \to \Omega^{\bullet}(1/f)$ is a quasi-isomorphism.

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Freeness is computable (related to Quillen-Suslin Th.). Spencer property is computable (with Groebner basis in A_n).

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The class (Spencer + free) strictly contains

- all non constant $f(x,y)$ (K. Saito; F. Calderón) and
- all free arrangement of hyperplanes in $\mathbb{C}^n$ (for $n \in \mathbb{N}$) (F. Calderón-L. Narváez).

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$f = xyz(x + y)(x + z)(y + z)(x + y + z)$ if free and Spencer.
$f = xyz(x + y + z)$ is Spencer but not free.
$f = (x + yz)(x^4 + y^5 + xy^4)$ is free but not Spencer (F. Calderón-L. Narváez).

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So $Ann^{(1)}(\frac{1}{f}) = Ann(\frac{1}{f})$.

Compute $Der(\log f)$ via $Syz(f'_x, f'_y, f'_z, f)$ (Groebner basis in $\mathbb{C}[x,y,z]$).

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So $Ann^{(1)}(\frac{1}{f}) = Ann(\frac{1}{f})$.

By a computation with `Macaulay2`, $Der(\log f)$ is generated by $\delta_1 = x\partial_x + y\partial_y + z\partial_z$

$$\delta_2 = y(x+y)(y+z)\partial_y - z(x+z)(y+z)\partial_z$$

$$\delta_3 = y(x-y)(x+y)\partial_y + z(x+z)(x+3y+3z)\partial_z$$

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Then (as announced some slides before)
$Ann^{(1)}(\frac{1}{f}) = Ann(\frac{1}{f})$ is generated by

$$P1 = x\partial_x + y\partial_y + z\partial_z + 7$$

$$P_2 = y(x+y)(y+z)\partial_y - z(x+z)(y+z)\partial_z + (y-z)(x+4y+4z)$$

$$P_3 = y(x-y)(x+y)\partial_y + z(x+z)(x+3y+3z)\partial_z + 3x^2 + 5xy - 4y^2 + 8xz + 8yz + 8z^2$$

A (personal) tautology

Homo sapiens invented the natural numbers ($\mathbb{N}$) to count things.

A (personal) tautology

When computations became hard to achieve *homo sapiens* invented Mathematics.

Computer Algebra is a powerful tool in Mathematics (and in particular in D-modules theory).

A (personal) tautology

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Modern Industrial Society needs to do big/heavy computations. In order to simplify them (and essentially –at least in D-module theory– all non trivial computation are heavy)

we must use meaningful and deep mathematical ideas and results.

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Modern Industrial Society needs to do big/heavy computations. In order to simplify them (and essentially –at least in D-module theory– all non trivial computation are heavy)

Testing equality $Ann^{(1)}(\frac{1}{f}) = Ann(\frac{1}{f})$ is a modest and clear example of such *tautology*.

Thank you very much.

References

References

Additional results

The following slides give more precise results
on the subject of the talk.

Free (hypersurfaces)

(K. Saito) $f \in \mathbb{C}[x]$ (non constant) defines a free hypersurface (in $\mathbb{C}^n$) if the module $Der(\log f)$ is a free $\mathbb{C}[x]$–module.

If so, we also say that f is free.

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(K. Saito) Any non constant polynomial in two variables $f(x, y)$ is free.

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$f = xyz(x + z + z)$ is not free.

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If so, we also say that f is free.

Freeness is computable (K. Saito's criterion + effective Quillen-Suslin).

LCT

(L. Narváez, D. Mond, F.J.C.J.) If $f = 0$ is a free and locally quasi-homogeneous hypersurface (in $\mathbb{C}^n$) then f satisfies LCP.

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So, for this class of f, by using Oaku-Takayama algorithm, $H^p(\Omega^\bullet(\log f)) = H^p(\Omega^\bullet(1/f))$ is computable for all p. So, for this class of f, we have a positive solution of Problem 4 (the cohomology of $\Omega^\bullet(\log f)$ is computable)

Free + Locally Quasi-homogeneous?

How big is the class

$\{f \in \mathbb{C}[x] \mid$ free $+$ locally quasi-homogeneous $\}$?

Free + Locally Quasi-homogeneous?

How big is the class
$\{f \in \mathbb{C}[x] \mid$ free $+$ locally quasi-homogeneous $\}$? Previous class strictly includes: a) all the free hyperplane arrangements.
b) all locally quasi-homogeneous plane curves $f(x, y) = 0$.

LCT for curves

(F.J. Calderón, L. Narváez, D. Mond, F.J.C.J.) If $f(x, y) = 0$ is a (reduced) plane curve then f satisfies LCP if and only if and all its singularities are quasi-homogeneous.

LCT for curves

(F.J. Calderón, L. Narváez, D. Mond, F.J.C.J.) If $f(x, y) = 0$ is a (reduced) plane curve then f satisfies LCP if and only if and all its singularities are quasi-homogeneous.

$f = x^4 + y^5 + xy^4 = 0$ has a non quasi-homogeneous singularity at the origin. Since f is free then f does not satisfy LCP. Since f is Spencer $Ann^{(1)}(\frac{1}{f}) \subsetneq Ann(\frac{1}{f})$.

Torelli's conjecture

Conjecture. For any nonzero polynomial $f \in \mathbb{C}[x]$, $Ann^{(1)}(\frac{1}{f}) = Ann(\frac{1}{f})$ if and only if $i_f : \Omega^\bullet(\log f) \to \Omega^\bullet(1/f)$ is a quasi-isomorphism.

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(J.M. Ucha-F.J.C.J.) If $f \in \mathbb{C}[x]$ is (Spencer + free) then previous conjecture is satisfied.